Gestion des stocks et de la production intégrant des retours de produits

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Introduction

L'objectif de cette thèse est de déterminer des règles de gestion pour les systèmes de production comportant des flux en provenance des clients vers le fabricant.

Dans cette introduction, nous présentons en premier lieu le contexte de la logistique inverse : les acteurs, leurs motivations, les étapes d'un retour et les décisions associées. Dans un deuxième temps, nous délimitons le cadre de nos recherches en donnant la problématique de cette thèse. Enfin nous annonçons le plan et les principales contributions.

1.1 La logistique inverse

En 2006, la quantité de déchets produits dans le monde est évaluée à plus de 3.4 milliards de tonnes. En France, la même année, la production de déchets ménagers est de 354 kg en moyenne par habitant selon l'ADEME (2009). Toutes ces quantités augmentent d'année en année.

> "Rien ne naît ni ne périt, mais des choses déjà existantes se combinent, puis se séparent de nouveau."
>
> *Anaxagore de Clazomènes, V^e siècle avant EC.*

Cette citation, reprise par Lavoisier sous sa forme populaire "Rien ne se perd, rien ne se crée, tout se transforme", retransmet notamment l'idée que les ressources terrestres sont en quantités limitées. Ainsi, celles-ci finiront nécessairement par décliner si nous continuons de les extraire. Dans ce contexte, la réutilisation, la réparation et le recyclage semblent être les seules solutions qui s'offrent à nous pour maintenir notre niveau de vie. Cette obligation à long terme fait souvent face à une réalité économique. Cependant ces deux aspects ne sont pas systématiquement en opposition. Il existe de nombreux cas industriels où la réutilisation, la réparation et le recyclage sont économiquement viables, par exemple les produits consignés ou d'occasion. La gestion de toutes ces activités visant à faire vivre plusieurs cycles à un produit ou à sa matière première s'inscrit dans un cadre plus large appelé logistique inverse.

1.1.1 Définition

Il existe plusieurs définitions de la logistique inverse. Par exemple celle proposée par Fleischmann (2000) :

> "Reverse Logistics is the process of planning, implementing, and controlling the efficient, effective inbound flow and storage of secondary goods and related information opposite to the traditional supply chain direction for the purpose of recovering value or proper disposal."

> "*La logistique inverse considère le processus de planification, de mise en œuvre et de contrôle de la performance, du flux entrant et du stockage des produits industriels et de leurs informations correspondantes, allant dans une direction opposée à la chaîne logistique traditionnelle, dans le but de récupérer de la valeur ou d'être éliminés convenablement.*"

Dans ce document nous retiendrons un périmètre plus large pour la logistique inverse, incluant les flux de retours (d'un client vers un fabricant), mais aussi leur impact sur la logistique traditionnelle (d'un fabricant vers un client). Nous considérons la logistique inverse comme la gestion des systèmes industriels comportant des flux de produits allant dans une direction opposée à la chaîne logistique traditionnelle.

1.1.2 Motivation des acteurs

Les principaux acteurs de la logistique inverse sont les clients et les fabricants, cependant d'autres acteurs peuvent intervenir. Ainsi, ce ne sera pas forcément le distributeur opérant pour le flux classique, qui s'occupera de la collecte ; les collecteurs pouvant par exemple être des entreprises spécialisées ou des collectivités. La figure 1.1 donne un aperçu des différentes parties opérationnelles de la chaîne. Pour ce qui est des acteurs de la couche management et décision, nous retrouvons des entités comme des gouvernements, des industriels, des vendeurs, et des recycleurs.

Le flux traditionnel est généralement créé par la loi de l'offre et de la demande. Dans le cas d'un flux de retour, les motivations des différents acteurs sont diverses. Elles diffèrent notamment en fonction de la catégorie à laquelle appartiennent ces acteurs. Le client est en début de la logistique inverse, c'est donc lui qui fait le premier pas. Si c'est un consommateur classique, les garanties, les réparations, la fin d'utilisation, la fin de vie du produit, ou encore une conscience écologique sont autant de raisons qui peuvent le pousser à renvoyer un produit. Notons que d'autres types de clients existent, dans le cas d'une usine cliente ou d'un maillon de la distribution, les motivations seront par exemple un défaut de qualité ou la réutilisation d'un élément de conditionnement. Du point de vue de l'entreprise réceptrice du produit, les trois principales motivations sont les suivantes :

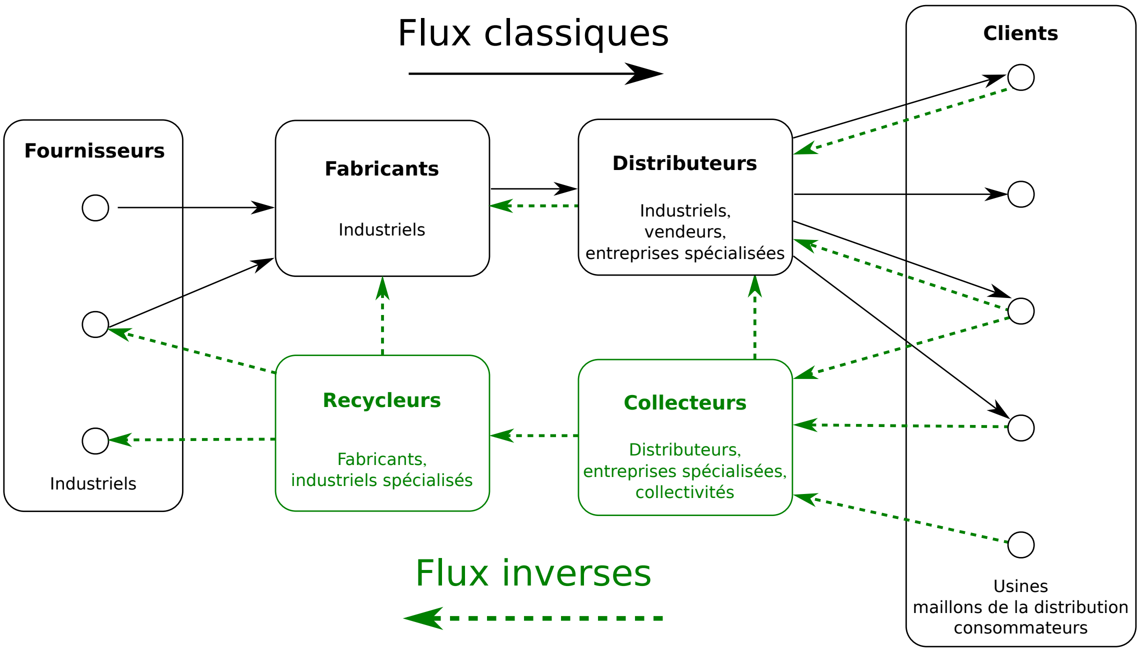


FIGURE 1.1 – Les acteurs de la logistique inverse (Fleischmann et al., 1997)

Un gain économique. Dans certains cas, la récupération des objets peut être économiquement viable. Notons par exemple les objets à fort niveau électronique qui arrivent souvent en fin de vie avec une grande valeur résiduelle.

Une obligation législative. Des lois récentes sur la responsabilité des industriels donnent lieu à de nouveaux flux en provenance du client. Ces obligations législatives tendent à se développer, notamment en Europe.

Une image "verte". La logistique inverse s'inscrivant dans le domaine du développement durable et de l'environnement, elle motive de plus en plus les industriels soucieux de leur image, c'est un argument marketing.

Dans la pratique ces trois arguments sont intimement liés. Par exemple, l'anticipation de la législation crée un gain économique, le respect de la législation donne une image verte et citoyenne, et l'image verte et citoyenne apporte des clients.

1.1.3 Caractéristiques des systèmes

Nous distinguerons deux grandes catégories de chaînes logistiques comportant des retours : les systèmes à boucle ouverte et les systèmes à boucle fermée (Fleischmann, 2000). Les premiers sont caractérisés par une réutilisation des retours visant à créer un produit différent et empruntant une autre chaîne de production/distribution que le produit initial, par exemple la réutilisation des pare-brises de voitures pour faire de la laine de verre[1].

[1] http://www.dictionnaire-environnement.com/pare-brise_vhu_ID3725.html

Comme dans cet exemple, les systèmes en boucle ouverte réutilisent souvent les produits à un niveau de valeur plus faible que le niveau initial. Dans les systèmes à boucle fermé, les retours sont utilisés pour créer des produits de même type, par exemple la réutilisation de tonner d'imprimante après remplissage. La distinction entre ces deux catégories n'est pas toujours évidente. Ainsi le recyclage du papier, collecté par les collectivités et revendu aux entreprises peut être considéré comme un système ouvert car les entreprises sont différentes et que le papier recyclé a une qualité inférieure au papier traditionnel ; il peut aussi être considéré comme une chaîne fermée car le papier ainsi recréé sera de nouveau collecté et recyclé de la même façon.

Une autre caractéristique des systèmes en logistique inverse est le niveau de réutilisation des produits (Landrieu, 2001). Ces niveaux sont décrits figure 1.2. Ils sont classés depuis le recyclage qui réutilise le retour au niveau de sa matière première, jusqu'à la réutilisation directe où l'on revend le produit sans aucune modification. A part cette dernière, les autres types de réutilisations nécessitent des étapes de désassemblage, de nettoyage, et de reconditionnement avant de pouvoir être inséré dans le flux traditionnel.

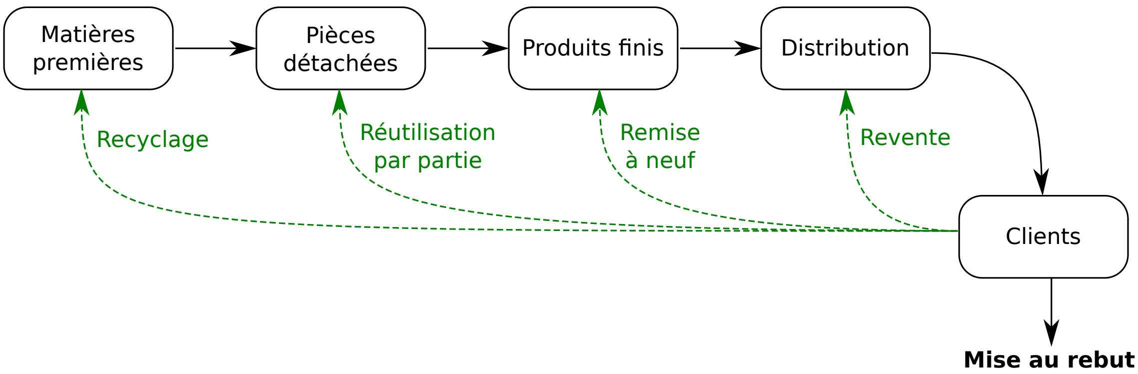


FIGURE 1.2 – Description des différents types de réutilisations

1.1.4 Problématiques étudiées dans cette thèse

Dans un système de production, l'encours est la quantité de produits en stock ou en train d'être produits. La tendance actuelle impose aux gestionnaires une diminution de l'encours pour libérer des capitaux et de l'espace. Ainsi, des retours non maîtrisés de produits finis, de sous-parties de produits ou encore de matières premières peuvent faire augmenter l'encours significativement. La gestion du flux de retour, son stockage et l'impact qu'il a sur la logistique traditionnelle est donc un enjeu important de la logistique inverse. Dans ce contexte, il semble essentiel pour une entreprise voulant réutiliser ses produits de gérer au mieux les flux de retours mais aussi les flux classiques impactés par les retours. Ce document s'inscrit dans cette démarche.

Nous nous intéressons à un problème opérationnel de gestion des stocks et de la production. Rappelons qu'il existe trois différents niveaux de décision dans une entreprise :

les décisions stratégiques qui engagent l'entreprise sur le long terme (le dimensionnement d'une usine, sa cadence, ou la localisation d'un entrepôt) ; les décisions tactiques qui engagent l'entreprise à moyen terme (l'agencement d'un atelier, le remplacement et le dimensionnement d'une machine, ou le design d'une référence de produit) ; et les décisions opérationnelles qui engagent l'entreprise à court et très court terme (le réapprovisionnement en pièce, les plannings de maintenance et de production, ou encore le prix de vente d'un produit). La gestion des stocks et de la production est donc un problème opérationnel. Il consiste notamment à mettre en service ou à arrêter les serveurs de production et, si le modèle le permet, décider de l'acceptation ou du rejet des retours à leur entrée dans le système. La figure 1.3 présente un exemple général reprenant les différentes réutilisations possibles décrites section 1.1.3.

Tous les modèles présentés dans ce document ont une capacité de production limitée. Notons que la capacité est une contrainte importante dans l'industrie relevant du niveau décisionnel tactique. Par exemple, une opération de nettoyage utilisant une machine traitant deux containers par cycle ne pourra pas avoir une cadence supérieure sans un changement de machine, soit un investissement conséquent. Cette contrainte incite les entreprises à anticiper la demande et à produire pour le stock de façon à prévenir la demande future.

Dans ce document nous modélisons les opérations de production comme des évènements discrets stochastiques. Les chaînes de montage automobile, les centres d'appels téléphoniques, le trafic routier, aérien et ferroviaire, sont des exemples de systèmes dynamiques complexes. Selon un certain point de vue, ils peuvent être spécifiés par des modèles à évènements discrets car leur activité est due à des occurrences d'évènements discrets. Certains évènements sont provoqués, comme par exemple le commencement d'une maintenance préventive, l'émission d'un appel téléphonique, ou l'autorisation de décoller pour un avion. D'autres sont subis, comme une panne, la réception d'un appel téléphonique, ou encore une grève du personnel. Ces évènements contrôlés ou non peuvent prendre un temps de réalisation très variable, ainsi une panne sur une même machine peut durer de quelques minutes à plusieurs jours suivant la disponibilité des pièces de rechange. Cette variabilité justifie la modélisation stochastique des systèmes de production.

Dans cette thèse, nous considérons un problème opérationnel de gestion de stock visant à satisfaire au mieux la demande et minimiser l'encours, tout en prenant en compte les flux de retours de produits. Nous nous plaçons dans un contexte où la capacité de production est limitée et où les aléas liés à la production ne sont pas négligeables. Les variables de décision sont liées à la mise en service ou l'arrêt des serveurs de production et de remise à neuf et, si le modèle le permet, l'acceptation des retours.

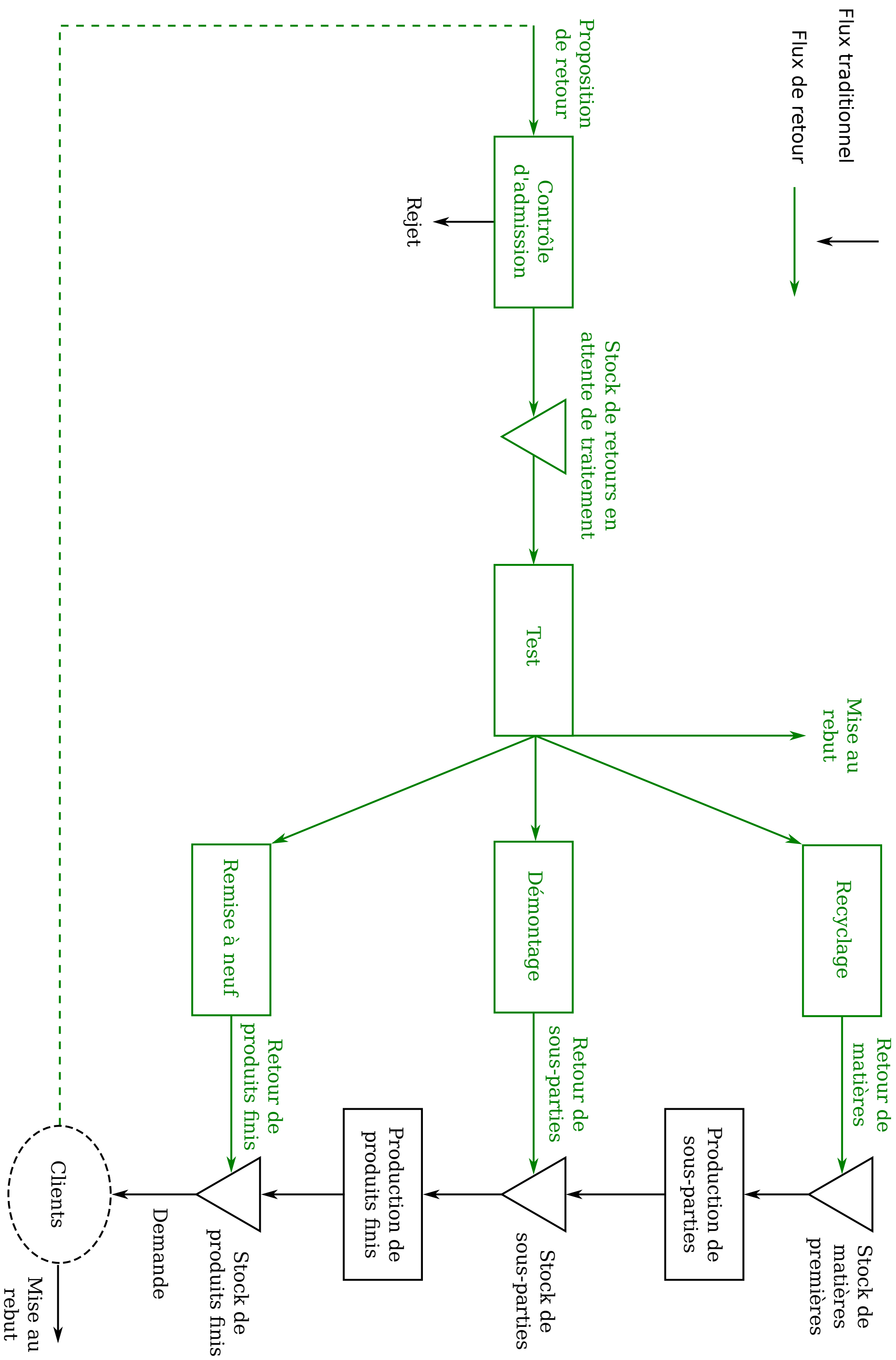


Figure 1.3 – Exemple présentant les phases d'un retour et un flux de production

1.2 Formulation mathématique adoptée : un exemple simple

Dans cette section, nous décrivons le type de modélisation retenu en présentant ses principales hypothèses, son critère de performance, ses variables de décision et les méthodes d'évaluation de la performance utilisées. Pour illustrer la formulation mathématique adoptée, nous proposons un exemple simple (voir figure 1.4), extrait de Veatch et Wein (1996), présentant un système à un étage de production sans retour de produit où les décisions prises visent à mettre en marche et arrêter la production. Les modèles présentés dans les chapitres suivants utilisent des notations et des formulations similaires à celles utilisées ici : processus sans mémoire, formulation en Processus de Décision Markovien (MDP), propriété de structure de la politique optimale et chaîne de Markov.

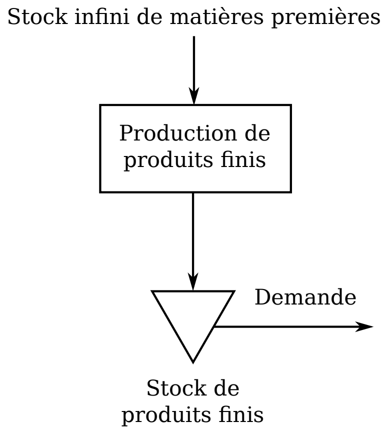


FIGURE 1.4 – Système à un étage de production sans retour de produit.

La modélisation continue des stocks semblant plus adaptée aux matières premières et aux liquides qu'aux produits industriels, nous considérons ici un espace d'état discret (i.e. produits indivisibles). De plus, nous adoptons une vision continue du temps. Dans les modèles à temps discontinu, l'état du système n'est connu qu'à certains instants prédéfinis. Au mieux, la prise de décision ne peut être faite qu'à ces instants. Nous considérons ici que l'information sur l'état du système est fiable et disponible en temps réel, et donc que les décisions peuvent être prises à n'importe quel moment. Cette hypothèse semble plausible au vu du développement des systèmes d'information dans les entreprises.

1.2.1 Modèle

Nous formulons ce problème comme un problème de file d'attente M/M/1 avec production par anticipation et demandes différées (voir figure 1.5). Ainsi, les produits sont fabriqués un par un et le délai de production est distribué selon une loi exponentielle de taux μ. Une fois fabriqués, les produits sont placés dans un stock où ils sont disponibles pour servir les demandes mais induisent un coût h par unité de produit et unité de temps. Les demandes arrivent une par une selon un processus de Poisson de taux λ. Si le stock est vide, les demandes sont mises en attente et le système encourt un coût de retard b par unité de temps et unité de demande. Le contrôle du système consiste à mettre en marche et arrêter la production, ces décisions étant possibles sans coût fixe et à tout instant (la préemption est autorisée). Le nombre de produits dans le stock à l'instant t est noté $X(t)$ et les demandes enregistrées sont modélisées par des valeurs négatives de $X(t)$. Remarquons que le taux de production μ doit être plus grand que le taux de demande λ pour que le système soit stable et que $X(t)$ ne diverge pas.

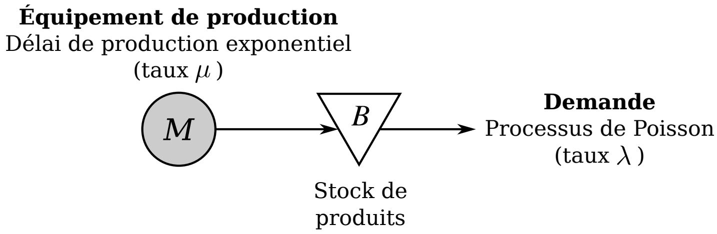


FIGURE 1.5 – File d'attente M/M/1 produisant par anticipation avec des retours Poissonniens

Pour chaque état du système, une politique π spécifie quand produire ou non. Les hypothèses faites (exponentialité des taux, coûts linéaires, préemption autorisée, etc...) permettent de décrire complètement l'état du système à l'instant t avec la variable $X(t)$. Avec une fonction de coût par unité de temps $c(x) = (h. \max\{x, 0\} + b. \min\{-x, 0\})$, un taux d'actualisation $\alpha > 0$ et un état initial x, l'espérance de coût actualisé du système est :

$$v_\alpha^\pi(x) = E\left[\int_0^{+\infty} e^{-\alpha t} c(X(t))\mathrm{d}t | X(0) = x, \pi\right].$$

Notre objectif est de déterminer la politique optimale, noté $\pi^\star$, qui minimise l'espérance de coût actualisé $v_\alpha^\pi(x)$ sur un horizon infini : $v_\alpha^\star(x) = \min_\pi v_\alpha^\pi(x)$. De plus, nous nous intéressons aussi à l'espérance de coût moyen du système :

$$g^\star = \min_\pi \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T c(X(t))\mathrm{d}t | X(0) = x, \pi\right].$$

1.2.2 Formulation en processus de décision markovien

Ce problème peut se formuler comme un processus de décision markovien (MDP, voir Puterman, 1994). Avec l'équation d'optimalité

$$v_\alpha^\star = \mathcal{T} v_\alpha^\star, \tag{1.1}$$

l'opérateur $\mathcal{T}$ est une fonction sur v telle que :

$$\mathcal{T} v(x) = \frac{1}{\lambda + \mu + \alpha}\left[c(x) + \lambda v(x-1) + \mu. \min\{v(x+1), v(x)\}\right].$$

Dans cette formulation, l'optimalité du choix est assurée par la minimisation : $\min\{v(x+1), v(x)\}$. Ainsi, la décision de produire sera prise si $v(x+1) < v(x)$; inversement, la décision de ne pas produire sera prise si $v(x+1) \geq v(x)$. Nous observons que la décision est fonction du signe de $\Delta v(x) = v(x+1) - v(x)$.

Notons que le coût moyen optimal satisfait l'équation d'optimalité suivante (Weber et Stidham, 1987) :

$$v_0^\star(x) + \frac{g^\star}{(\lambda + \mu)} = \mathcal{T} v_0^\star(x).$$

1.2.3 Caractérisation de la politique optimale

Nous considérons en premier lieu le critère de coût actualisé ($\alpha > 0$). Nous cherchons à démontrer que la politique optimale est une politique à seuil (à niveau de recomplètement).

Lemme 1.2.1. *L'opérateur $\mathcal{T}$ propage la convexité : si une fonction v est convexe alors la fonction $\mathcal{T}v$ sera convexe.*

Démonstration. La fonction de coût $c(x)$ est convexe. Si l'on suppose que v est convexe, $v(x-1)$ est convexe, et $\min\{v(x), v(x+1)\}$ est convexe (Koole, 2006). L'opérateur $\mathcal{T}$ propage donc la convexité car la fonction $\mathcal{T}v$ est une combinaison convexe de fonction convexe. □

Étant donné la propriété de contractance de l'opérateur $\mathcal{T}$ (Koole, 2006), toute suite $v_\alpha^{n+1} = \mathcal{T} v_\alpha^n$ converge vers la solution unique $v_\alpha^\star$ de l'équation d'optimalité (1.1) :

$$v_\alpha^\star = \lim_{k \to \infty} v_\alpha^k, \text{ avec } v_\alpha^{k+1} = \mathcal{T} v_\alpha^k, \ \forall v_\alpha^0.$$

La relation précédente et le lemme 1.2.1 nous permettent de démontrer par récurrence la structure de la politique optimale :

Theorem 1.2.2. *La politique optimale est une politique à seuil $S_\alpha^\star$, telle qu'il est optimal de produire si le niveau de stock est plus petit que $S_\alpha^\star$ et ne pas produire sinon.*

Démonstration. Avec $v_\alpha^0(x) = 0 \ \forall x$ (qui est convexe), tout terme de la suite $v_\alpha^{n+1} = \mathcal{T} v_\alpha^n$ est convexe car $\mathcal{T}$ propage la convexité (Lemme 1.2.1). Donc la limite de la suite $v_\alpha^\star$ est convexe et il existe un niveau $S_\alpha^\star$ tel que :

$$\begin{cases} \Delta v_\alpha^\star(x) > 0 & \forall x < S_\alpha^\star, \\ \Delta v_\alpha^\star(x) \geq 0 & \forall x \geq S_\alpha^\star. \end{cases}$$

□

Pour le critère de coût moyen, la politique optimale est la limite de la politique optimale en coût actualisé quand α tend vers 0 (Weber et Stidham, 1987). De plus $g^\star = \alpha v_\alpha^\star(x) \ \forall x$.

1.2.4 Expression du seuil optimal

Nous dénombrons trois grandes familles d'outils pour évaluer la performance des modèles de systèmes de production à évènements discrets stochastiques : la simulation, les méthodes numériques et les méthodes analytiques. La simulation n'est pas utilisée dans ce document. Elle consiste à simuler le fonctionnement du système grâce à de nombreux tirages aléatoires permettant de reproduire son comportement supposé. Cette méthode a l'avantage de pouvoir étudier des systèmes très complexes et de ne pas contraindre le modèle à certaines distributions de probabilité. Dans ce document nous utilisons la formulation en MDP qui permet de déterminer numériquement les performances du système par itération sur la valeur (voir section précédente), cependant cette méthode impose au processus d'être sans mémoire, ce qui impose des distributions exponentielles (ou des combinaisons d'exponentielles) et contraint à ne considérer que des systèmes de petites tailles pour limiter l'espace d'état. Dans certains cas très particuliers et/ou très simples, il existe des solutions analytiques aux problèmes considérés. Ces résultats sont généralement possibles lorsque le système peut être modélisé comme une chaîne de Markov monodimentionnelle (composé d'une seule variable). D'autres résultats très généraux peuvent être obtenus grâce à des arguments d'échanges ou de trajectoires.

Dans notre exemple, la politique optimale est une politique à seuil, nous pouvons donc écrire la chaîne de Markov associée comme un processus de naissance et de mort de taux $\rho = \lambda/\mu$ (voir figure 1.6).

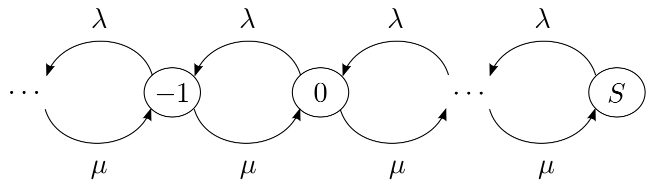


FIGURE 1.6 – Chaîne de Markov de la file d'attente M/M/1 avec enregistrement de la demande.

L'expression de la chaîne de Markov étant suffisamment simple, nous pouvons dériver ses probabilités stationnaires :

$$P_x = \begin{cases} (\rho)^{x-1}(1-\rho) & \text{si } x \leq S, \\ 0 & \text{sinon,} \end{cases}$$

et déterminer le coût moyen $g(S)$:

$$g(S) = \sum_{x \in \mathbb{Z}} (hP_x \max\{x, 0\} + bP_x \max\{-x, 0\}) = h\left(S + \frac{\rho(\rho^S - 1)}{1-\rho}\right) + b\frac{\rho^{S+1}}{1-\rho}.$$

Enfin nous déterminons le niveau de base stock optimal (en coût moyen) en résolvant l'équation $S^\star = \min\{S : \Delta g(S) \geq 0\}$:

$$S^\star = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \rho} \right\rfloor .$$

Pour le critère de coût actualisé, les calculs sont plus compliqués et font intervenir des transformées de Laplace (Dusonchet et Hongler, 2003), cependant il existe une expression analytique du coût et du seuil optimaux.

1.2.5 Effet des paramètres sur la politique optimale

Dans cette section, nous étudions l'influence des paramètres sur la politique optimale. Dans notre exemple nous pouvons facilement conclure que $S^\star$ est croissant en b, λ et décroissant en h, μ. Cependant cette méthode nécessite de connaître l'expression analytique de la politique optimale, ce qui n'est que très rarement le cas.

Plus généralement nous pouvons poser le problème comme ceci : pour connaître l'impact d'un paramètre p (par exemple $p = \lambda$) sur la politique optimale, nous pouvons nous intéresser à l'évolution de $\Delta v(x,p)$ quand p varie, nous cherchons pour quel signe de ϵ nous avons

$$\Delta v(x, p + \epsilon) \geq \Delta v(x, p).$$

Par exemple, si $\epsilon \geq 0$ alors le seuil S est décroissant en p. Cette propriété peut être montrée par récurrence (Çil et al., 2009), tout comme la convexité dans la section 1.2.3.

Notons que cette thèse porte principalement sur des modèles où la chaîne de Markov est multidimensionnelle, ce qui rend généralement inaccessible l'expression de ses probabilités stationnaires et donc les résultats analytiques qui en découlent.

1.3 Plan et contributions

Ce document est organisé par article. Chacun des chapitres 3 à 6 peut être lu indépendamment des autres.

Le chapitre 2 présente un état de l'art sur les modèles stochastiques de gestion de stock avec des retours de produits. Il se décompose en trois parties : les modèles monodimentionnel, les modèles avec un stock de produit à remettre à neuf modélisé explicitement et les modèles avec plusieurs étapes de production ou d'assemblage.

Dans le chapitre 3 nous considérons un problème de réutilisation de produits comme produits finis ou comme produits en cours de production (voir exemple figure 1.7). Par exemple, dans le cas d'un produit renvoyé pour insatisfaction[2] si l'emballage est intact, il pourra être vendu directement, dans le cas contraire, un reconditionnement préalable sera nécessaire. En pratique, nous modélisons un système de production et de stockage à deux

[2] Dans le droit européen, un client dispose de 7 jours ouvrables pour annuler un achat effectué sur internet s'il ne convient pas à ses attentes, même s'il n'est pas défectueux.

étages avec des capacités de production limitées modélisées par les serveurs exponentiels. Le stock final (en aval) fait face à une demande poissonnienne. Chaque stock reçoit des retours de produits, arrivant selon des processus de Poisson indépendants. Ces retours peuvent être utilisés pour répondre à la demande. L'objectif est de contrôler la production afin de minimiser le coût actualisé (ou moyen) de stockage et de rupture de stock. Pour le problème à un seul étage, nous caractérisons complètement la politique optimale. Nous montrons qu'elle est à seuil et nous déduisons une formule explicite de ce seuil. Pour le problème à deux étages, nous montrons que la politique optimale est caractérisée par des courbes de commutation et nous montrons des propriétés de structure sur ces courbes. Dans une étude numérique, nous étudions trois politiques heuristiques : la politique base stock, la politique Kanban et la politique fixed buffer. Cette dernière obtient des résultats médiocres, tandis que les performances relatives des deux autres dépendent de la position du goulet d'étranglement. Nous montrons aussi que les taux de retours ont un effet non monotone sur les coûts et une forte incidence sur les performances des heuristiques. Enfin, nous observons que la réception des retours à l'étage amont peut être préférable à la réception à l'étage aval dans certaines situations.

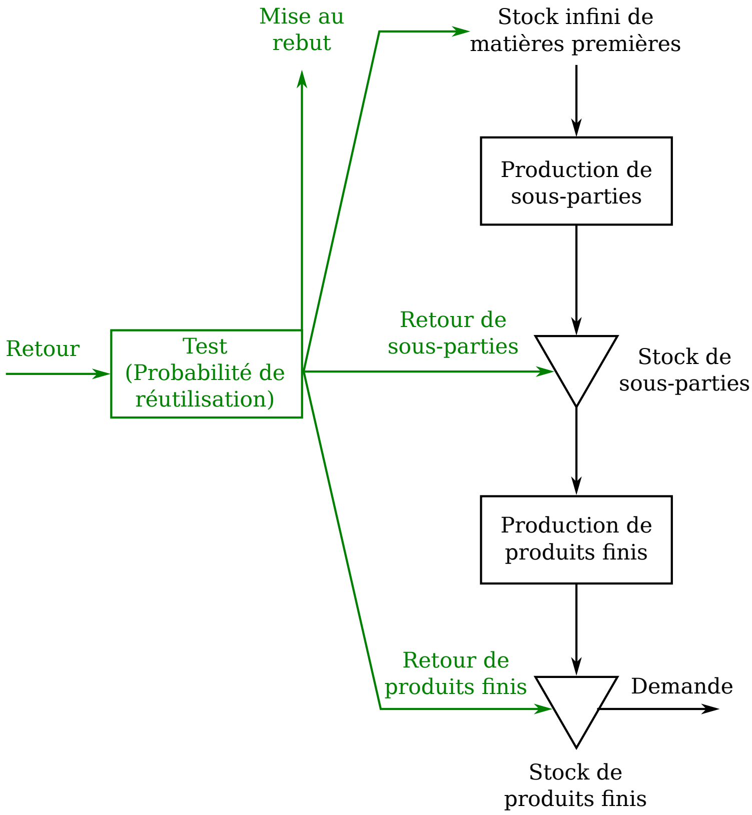


Figure 1.7 – Réutilisation par partie et revente directe

Le chapitre 4 présente un problème où les retours peuvent être refusés ou acceptés par l'entreprise. Après acceptation, ils nécessitent une remise à neuf pour être vendus comme un nouveau produit. (voir exemple figure 1.8). Un exemple est la réutilisation de photocopieurs qui peuvent être revendus (ou reloué) après réparations. En pratique nous nous intéressons à un problème hybride de production et de remise à neuf. Nous modélisons la production et la remise à neuf par des serveurs uniques avec des temps de traitement exponentiellement distribués. Les demandes des clients et les produits retournés arrivent dans le système en fonction de processus de Poisson indépendants. Un produit retourné peut être soit rejeté ou accepté. Une fois accepté, il est placé dans un stock en attente d'une remise à neuf. Les nouveaux produits et les produits remis à neuf sont placés dans un stock de produits finis faisant face aux demandes des clients. Les coûts suivants sont inclus dans le modèle : le stockage, la rupture de stock, la production, la remise à neuf, l'admission et le rejet. Nous montrons que la politique optimale est caractérisée par des courbes de commutation pour la production, la remise à neuf et le contrôle d'admission des retours. Nous obtenons des résultats de monotonie pour ces courbes de commutation. Pour finir, nous créons une politique heuristique à partir des résultat précédents, nous adaptons plusieurs politiques heuristiques trouvées dans la littérature à notre modèle et nous réalisons une étude numérique afin de comparer leurs performances à celles de la politique optimale.

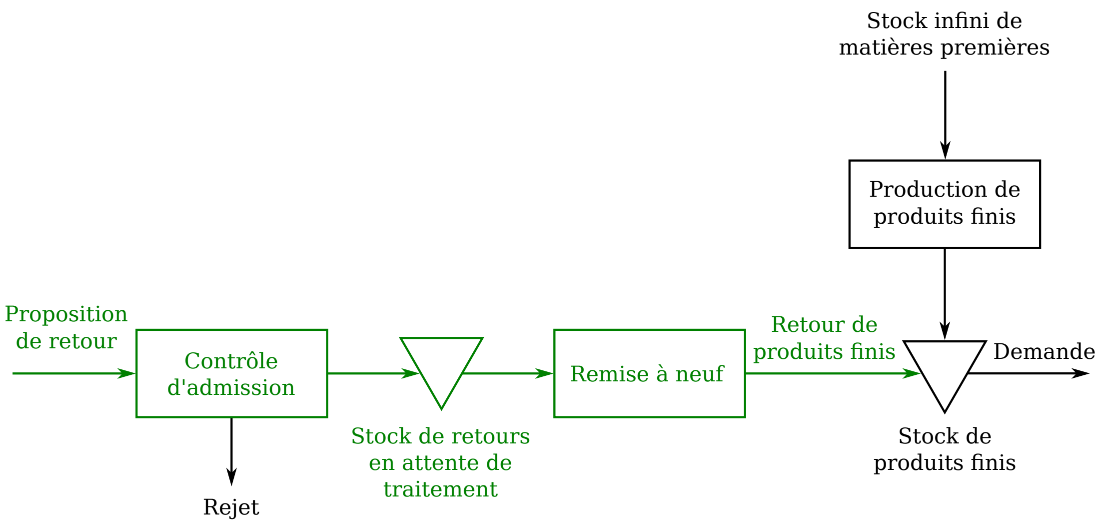


FIGURE 1.8 – Remise à neuf

Dans le chapitre 5 nous présentons un système où les clients préviennent à l'avance de l'envoi de leurs produits (voir exemple figure 1.9). Une entreprise peut, par exemple, imposer à ses clients de prévenir du renvoi pour insatisfaction ou garantie plusieurs jours à l'avance. Nous considérons ainsi un système de production et de stockage incluant les retours de produits annoncés à l'avance par les clients. Les demandes et les annonces de retours se produisent en fonction de processus de Poisson indépendants. Un retour annoncé est soit effectivement retourné soit annulé après un délai aléatoire distribué selon une loi Erlang. En cas de rupture de stock, nous considérons successivement les cas de ventes perdues et d'enregistrement de la demande. En utilisant une formulation en processus de décision de markovien, nous caractérisons la politique de production minimisant le coût actualisé (ou le coût moyen) sur un horizon infini, dans le cas où les retours sont annoncés et dans le cas où ils ne le sont pas. Numériquement, nous donnons un aperçu de la valeur de cette information. Enfin, nous considérons le cas combinant une information avancée sur les retours et sur les demandes. Ce travail a été effectué dans la continuité de la thèse d'Hichem Zerhouni (soutenue en 2009). Un partie de l'étude numérique présentée est tirée de cette thèse.

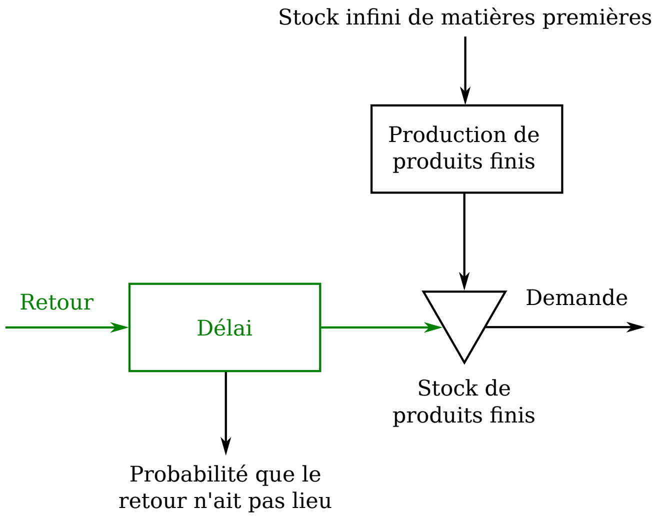


FIGURE 1.9 – Information avancée sur les retours

Le chapitre 6 s'intéresse à une formulation générique des problèmes précédents. Il fournit un cadre général pour étudier l'effet des paramètres d'un système sur la politique optimale de ce même système (voir section 1.2.5). Nous formulons un problème général en MDP à l'aide de deux types d'opérateurs : les opérateurs de choix et les opérateurs de translation. Nous étudions dans quels cas ceux-ci permettent un effet monotone des paramètres du système sur la politique optimale, c'est-à-dire quand ces opérateurs propagent des propriétés de type supermodularité. Deux exemples sont proposés pour illustrer notre étude. Enfin une extension est proposée permettant d'étendre nos résultats et certains résultats de la littérature aux opérateurs dont le taux de service est fonction de l'état du système.

La conclusion de cette thèse est présentée dans le chapitre 7 où des perspectives de recherches sont proposées.

Chapitre 2

État de l'art

Dans ce chapitre, nous nous intéressons aux principaux modèles stochastiques de gestion de stock avec un flux de production et un flux de retour. Nous décomposons notre état de l'art en fonction de la structure du réseau modélisé. La section 2.1 concerne les modèles à un étage où les retours peuvent être réutilisés directement. La section 2.2 se focalise sur les modèles où les retours doivent être remis à neuf avant réutilisation. Enfin, la section 2.3 s'intéresse aux modèles avec plusieurs étages d'approvisionnement où les retours arrivent à un ou plusieurs étages. Notons que le terme approvisionnement désigne indistinctement une étape de transport ou de fabrication. Une attention particulière est portée à la structure et à l'optimalité des politiques proposées, ainsi qu'à la modélisation ou non de la capacité d'approvisionnement.

2.1 Réutilisation directe

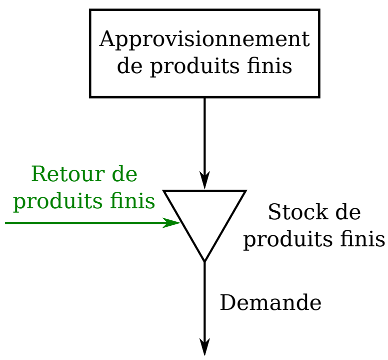


Figure 2.1 – Modèle avec réutilisation directe des retours

Dans cette section, nous nous intéressons aux modèles de la littérature ayant un seul stock, où l'approvisionnement en nouvelles pièces se fait en une étape, et où les retours sont réutilisables comme produits finis (voir figure 2.1) sans étape intermédiaire. Le tableau 2.1 détaille les hypothèses de chaque modèle présenté dans cette section. Dans ce tableau, la variable I (dépendant du temps) représente la position de stock, soit la position du stock physique de produits finis, moins le nombre de produits demandés en attente, plus le nombre de produits commandés en cours d'approvisionnement.

A notre connaissance, le premier article traitant d'un problème à un étage avec des retours est celui de Simpson (1970) qui modélise un système à temps discret en supposant des distributions quelconques pour la demande et les retours. Les demandes non satisfaites sont différées (mise en attente), les coûts sont linéaires et les retours ne peuvent pas être rejetés. Le délai de remise à neuf est négligé et donc non modélisé explicitement. Au sujet de l'approvisionnement, le délai est stochastique et la capacité est supposée infinie. Une politique (S_m) est proposée pour la fabrication. Cette politique consiste à approvisionner la quantité nécessaire pour que la position de stock atteigne S_m à chaque période. La simulation est utilisée pour évaluer le paramètre S_m de façon optimale.

Avec un coût fixe de commande, Fleischmann et al. (2002) considère un modèle à temps continu. En modélisant des demandes et des retours poissonniens et un délai d'approvisionnement constant, ils prouvent que la politique (s_m, Q_m) est optimale avec Q_m la taille des lots commandés quand la position de stock devient inférieure ou égale à s_m.

Fleischmann et Kuik (2003) étendent ce dernier modèle à des demandes et des retours de distribution quelconque. Les retours sont modélisés comme une demande négative. L'optimalité en coût moyen de la politique (s_m, S_m) est prouvée pour les cas, temps discret et temps continu. Cette politique consiste à recompléter la position de stock jusqu'à S_m quand celui-ci est inférieur ou égal à s_m. Notons que la méthode consistant à considérer une demande négative pour modéliser les retours est utilisée dans plusieurs articles, notamment par Beltran et Krass (2002) qui proposent une adaptation du modèle de Wagner et Whitin (1958) (Dynamic Lot Sizing, DEL) avec des retours dans un cadre déterministe.

Les modèles précédents sont tous des modèles de stockage pur (pure inventory) où la capacité d'approvisionnement est infinie. Une autre classe de modèle est la classe de production et de stockage (production / inventory) qui représente explicitement la production et sa capacité. Les files d'attente (Buzacott et Shanthikumar, 1993 ; Zipkin, 2000) permettent notamment de modéliser ce comportement en contraignant les tailles des lots produits et en rendant aléatoire les délais de production.

Peu de travaux modélisent les retours avec des files d'attentes. C'est le cas de celui présenté par Gayon (2006) qui pose un problème à un étage avec des retours arrivant unité par unité. Les produits sont fabriqués un par un avec un temps de production exponentiellement distribué. Les retours et les demandes sont Poissonniens et indépendants l'un de l'autre (ces hypothèses correspondent à une file d'attente M/M/1 produisant par anticipation avec des retours poissonniens). L'optimalité en coût moyen et en coût actualisé de la politique base stock est démontrée. De plus, une formule analytique du coût est proposée.

Étendant ce précédent modèle, Zerhouni et al. (2009) modélisent des retours pouvant être acceptés comme produits finis ou rejetés avec un coût c_b. Dans les cas de vente perdue et vente différée, ils montrent que la politique (S_m, s_d) est optimale (produire quand le

	Simpson, 1970	Fleischmann et al., 2002	Fleischmann et Kuik, 2003	Gayon, 2006	Zerhouni et al., 2009	Zerhouni et al., 2010	Zerhouni, 2009
Temps	Discret	Continu	Discret/continu	Continu	Continu	Continu	Continu
Horizon	Infini	Infini	Infini	Infini	Infini	Infini	Infini
Critère de coût	Moyen	Moyen	Moyen	Moy./actualisé	Moy./actualisé	Moy./actualisé	Actualisé
Demande							
- Vente	Différée	Différée	Différée	Perdue	Diffé./Perdue	Perdue	Perdue
- Distribution	Générale	Poisson	Générale	Poisson	Poisson	Poisson	Poisson
Retour							Délai expo.
- Distribution	Générale	Poisson	Générale	Délai expo.	Poisson	Délai expo.	Poisson
- Lien avec la demande	Indépendant	Indépendant	Indépendant	Probabilité	Indépendant	Probabilité	Indépendant
- Rejet autorisé	Non	Non	Non	Non	Oui	Non	Non
- Décision de rejeter					Si $I \geq s_d$		
Approvisionnement							
- Capacité	Infinie	Infinie	Infinie	Prod. 1/1	Prod. 1/1	Prod. 1/1	Prod. 1/1
- Coût fixe	Non	Oui	Oui	Non	Non	Non	Non
- Délai / temps de service	Général	Constant	Général	Exponentiel	Exponentiel	Exponentiel	Exponentiel
- Décision d'approvisionner	Si $I < S_m$	Si $I \leq s_m$	Si $I \leq s_m$	Si $I < S_m$	Si $I < S_m$	(Complexe)	(Complexe)
- Taille de lot	Jusqu'à S_m	Q_m	Jusqu'à S_m	1	1	1	1
Politique	(S_m)	(s_m, Q_m)	(s_m, S_m)	(S_m)	(S_m, s_d)	(Complexe)	(Complexe)
- Optimalité démontrée		Oui	Oui	Oui	Oui	Oui	Oui

TABLE 2.1 – Modèles à un étage avec des retours

stock est inférieur à S_m et rejeter les retours quand le stock dépasse s_d). De plus, ils montrent des relations d'ordre entre les paramètres optimaux S_m et s_d en fonction du coût c_b. Enfin, dans le cas de vente différée, ils obtiennent une expression analytique du niveau de recomplètement optimal S_m en fonction de $(s_d - S_m)$. Notons que ce modèle est détaillé dans la section 4.5 de ce document.

Le modèle de Gayon (2006) est généralisé dans une autre direction par Zerhouni et al. (2010) qui considèrent des retours corrélés avec la demande antérieure. Une demande engendre un retour avec une certaine probabilité, et celui-ci arrive après un délai exponentiel. Dans le cas où le nombre de retours en cours est observable, la politique optimale est caractérisée. De plus, ce papier fourni une revue de littérature sur les travaux modélisant une corrélation entre la demande et les retours.

Le dernier modèle de file d'attente à un étage que nous connaissions est celui traité dans la thèse de doctorat de Zerhouni (2009). Il traite de l'information avancée sur les retours, où un retour annoncé a une probabilité d'être réalisé après un délai exponentiel. La structure de la politique optimale est caractérisée et le gain lié à l'information est testé numériquement. Cette étude est largement étendue dans le chapitre 5. Notons que Zerhouni (2009) fournit un état de l'art sur les modèles avec des retours corrélés à la demande antérieure. Ce type de modèles n'étant pas étudié dans notre document nous ne détaillerons pas plus avant ces modèles.

2.2 Remise à neuf

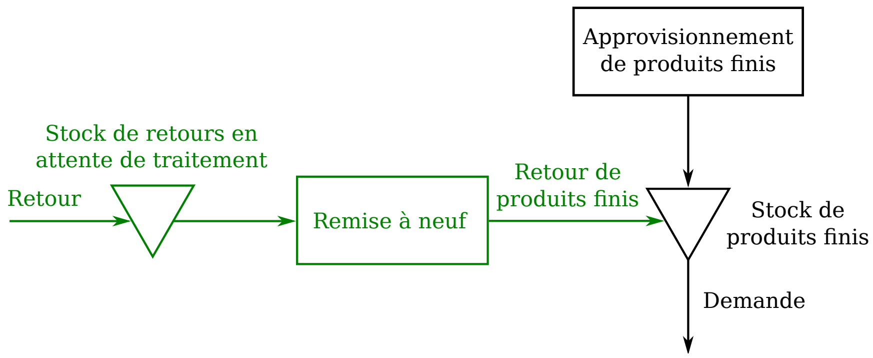


FIGURE 2.2 – Modèle avec remise à neuf des retours

Dans cette section, nous détaillons les principaux modèles où l'étape de remise à neuf est explicitement modélisée (voir figure 2.2). Le tableau 2.2 détaille les hypothèses de chaque modèle présenté dans cette section. Dans ce tableau, la variable I représente la position de stock de produits finis et la variable R le stock de retours pouvant être remis à neuf.

Le premier modèle considérant un stock de produits à remettre à neuf n'est pas un modèle stochastique, mais une extension du problème de quantité économique de commande (EOQ) proposé par Schrady (1967). Le premier modèle stochastique avec un stock explicite de retours à remettre à neuf est proposé par Simpson (1978). Il considère un modèle à temps discret où les délais d'approvisionnement L_M et de remise a neuf L_R sont constants et égaux ($L_M = L_R$) ; et avec la possibilité de stocker les retours avant leur remise à neuf, Simpson (1978) prouve que la politique optimale est une politique à trois paramètres (S_m, s_d, S_R), avec S_R le niveau de complètement pour la remise à neuf. Le modèle autorise le rejet des retours après acceptation (une fois qu'ils sont dans le stock), cependant la politique optimale n'utilise jamais cette option et effectue le rejet uniquement à l'arrivée des retours.

Inderfurth (1997) traite un cas similaire au modèle précédent où les retours acceptés sont directement remis à neuf sans attendre. Il montre que si le délai de remise à neuf est plus petit d'une unité que le délai de fabrication ($L_R + 1 = L_M$), une politique complexe avec trois paramètres (S_m, s_d, s'_d) est optimale.

Kiesmüller (2003) étudie un modèle similaire à celui d'Inderfurth (1997) mais avec des délais non égaux et sans option de rejet. En fonction de l'écart entre les délais, elle propose d'utiliser deux politiques heuristiques différentes. La spécificité de ces heuristiques est qu'elles ne prennent pas en compte tout l'historique des commandes passées et non arrivées mais juste une partie dans le calcul de la position de stock. Ces deux heuristiques sont testées et comparées avec un calcul de position de stock traditionnel.

Avec un délai de remise à neuf constant et la possibilité de rejeter les retours à leur arrivée, Heyman (1977) considère le premier modèle à temps continu avec retour. La demande et les retours sont poissonniens et indépendants les uns des autres, les coûts sont linéaires, l'approvisionnement en nouveaux produits est possible avec un délai nul et les retours acceptés sont directement remis à neuf. Ces hypothèses fortes impliquent que la politique optimale de gestion de stock est une simple politique (s_d). L'auteur fournit également une expression analytique pour s_d.

Avec un coût fixe de commande et un délai de production non nul mais sans possibilité de rejet des retours, Muckstadt et Isaac (1981) étendent ce modèle pour étudier un problème de location de photocopieurs. Avec le critère de coût moyen, une politique conventionnelle (s_m, Q_m) est testée, avec Q_m la taille des lots commandés quand le stock net devient inférieur ou égal à s_m. Les valeurs s_m et Q_m sont déterminées grâce à une approximation sur le niveau de stock. Notons qu'avec un délai de remise à neuf nul, Fleischmann et al. (2002) prouvent que la politique (s_m, Q_m) est optimale.

Avec une vision continue du temps, de nombreuses politiques sont présentées et comparées entre elles. Cependant aucune référence n'étudie leur optimalité. C'est le cas de van der Laan et Teunter (2006) qui étendent le modèle de Muckstadt et Isaac (1981) en ajoutant un stock de retours, un coût fixe pour la remise à neuf, des distributions générales pour la demande et les retours. Cependant ils limitent leur étude à des délais égaux et constants. Deux politiques sont proposées, la première contrôlant la remise à neuf en flux poussé et la deuxième contrôlant la remise à neuf en flux tiré. Une formule approchée pour les paramètres est comparée aux paramètres optimaux dans une étude numérique, mais

	Simpson, 1978	Inderfurth, 1997	Kiesmuller, 2003	Heyman, 1977	Muckstadt et Isaac, 1981
Temps	Discret	Discret	Discret	Continu	Continu
Horizon	Fini	Fini	Fini	Infini	Infini
Critère de coût	Total	Total	Moyen	Moy./actualisé	Moyen
Demande					Différée
- Vente	Différée	Différée	Différée	Différée	
- Distribution	Générale	Générale	Générale	Poisson	Poisson
Retour					
- Distribution	Générale	Générale	Générale	Poisson	Poisson
- Vs. demande	Indépendant	Indépendant	Indépendant	Indépendant	Indépendant
- Stockage	Oui	Non	Oui	Non	Non
- Rejet autorisé	Oui	Oui	Non	Oui	Non
- Rejet post-accept.	Oui				
- Décision de rejeter	Si $I + R \geq s_d$	Si $I \geq s_d$		Si $I \geq s_d$	
Approvisionnement					
- Capacité	Infinie	Infinie	Infinie	Infinie	Infinie
- Coût fixe	Non	Non	Non	Non	Oui
- Délai (Lm)	Constant = Lr	Constant = Lr	Constant	0	Constant
- Décision d'appro.	Si $I + R < S_m$	Si $I < S_m$ Jusqu'à S_m	a : Si $f(I) < S_m$ b : Si $I + R < S_m$	Si $I < 0$ Jusqu'à 0	Si $I \leq s_m$ Q_m
Remise à neuf	Oui	Oui	Oui	Oui	Oui
- Capacité	Infinie	Infinie	Infinie	Infinie	Infinie
- Coût fixe	Non	Non	Non	Non	Non
- Délai (Lr)	Constant = Lm	Constant = Lm	Constant	Constant	Constant
- Stock	Oui		Oui		
- Décision de r. à n.	Si $I < S_r$		a : Si $I < S_r$ b : Si $g(I) < S_r$		
Politique	(S_m, s_d, S_r)	(S_m, s_d)	a, b : (S_m, S_r)	(s_d)	(s_m, Q_m)
Optimalité	Oui	Oui	Non	Oui	

TABLE 2.2 – Modèles avec remise à neuf des retours

Van der Laan et Teunter, 2006	Van der Lann et al., 1996 a,b	Van der Laan et Salomon, 1997	a : Gupta et Korugan, 2000 b : Korugan et Gupta, 1998
Continu	Continu	Continu	Continu
Infini	Infini	Infini	Infini
Moyen	Moyen	Moyen	Moyen
Différée	Différée	Différée	a : Perdue
			b : Différée
Générale	Poisson	Coxian-2	Poisson
Générale	Poisson	Coxian-2	Poisson
Indépendant	Indépendant	Probabilité	Indépendant
Oui	Oui	Oui	Oui
Non	Oui	Oui	Oui
	Non	Non	Non
	a : Si $I+R<s_d^1$ et $R<s_d^2$	a : Si $I \geq s_d$	a : Fixed Buffer
	b : Si $R \geq s_d$	b : Si $R \geq s_d$	b : Kanban
Infinie	Infinie	Infinie	Prod. 1/1
Oui	Oui	Oui	Non
Constant = Lr	Constant	Constant	Exponentiel
Q_m Si $I \leq s_m$	Q_m si $I+R \leq s_m$	Q_m si $I \leq s_m$	a : Fixed Buffer
			b : Kanban
Oui	Oui	Oui	2 étapes en série
Infinie	Finie (C serveurs)	Infinie	Prod. 1/1
Oui	Non	Oui	Non
Constant = Lm	Exponentiel	Constant	Exponentiel
Oui	Oui (b : sans coût de stock.)	Oui	Oui (x2)
a : Q_r (flux poussé)	Push	a : Q_r (flux poussé)	a : Fixed Buffer
b : Q_r Si $I \leq s_r$		b : Jusqu'à S_r si $I \leq s_r$	b : Kanban
a : (s_m, Q_m, Q_r)	a : (s_m, Q_m, s_d^1, s_d^2)	a : (s_m, Q_m, Q_r, s_d)	a : Fixed Buffer
b : (s_m, Q_m, s_r, Q_r)	b : (s_m, Q_m, s_d)	b : $(s_m, Q_m, s_r, S_r, s_d)$	b : Kanban
Non	Non	Non	Non

aucune des deux politiques n'est démontrée optimale.

De la même façon, mais avec une capacité de remise à neuf limitée, van der Laan et al. (1996a,b) proposent un modèle où les délais d'approvisionnement et de remise à neuf sont respectivement constants et exponentiels. La demande et les retours sont indépendants et suivent des processus de Poisson, de plus, les retours peuvent être rejetés. Deux nouvelles politiques et deux types d'approximations sont proposées et comparées avec l'approximation présentée par Muckstadt et Isaac (1981). Notons que l'influence de la capacité de remise à neuf n'est pas étudiée.

Dans le cas où une demande crée un retour avec une certaine probabilité, van der Laan et Salomon (1997) proposent deux politiques généralisant les politiques de van der Laan et al. (1996b) et van der Laan et Teunter (2006). Lors d'une étude numérique, ils comparent plusieurs de ces politiques entre elles. Cette étude numérique est complétée par Teunter et Vlachos (2002).

Dans des notes courtes, Korugan et Gupta (1998) et Gupta et Korugan (2000) envisagent une modélisation en files d'attente pour un processus de remise à neuf en deux étapes remplissant un stock de produits finis. Le processus est modélisé sans mémoire (délais exponentiels et processus Poissonniens). Les politiques Kanban et fixed buffer sont testées indépendamment et les paramètres optimaux de ces politiques ne sont pas déterminés.

Notons que certains travaux modélisent des retours n'ayant pas la même qualité que les produits finis, même après remise à neuf (Kleber et al., 2002; Inderfurth, 2004). D'autres modélisent des ressources partagées entre la production de nouveaux produits et la remise à neuf (Teunter et al., 2008 ; Francas et Minner, 2009). Ces types de modèles n'étant pas traités dans ce document nous renvoyons à leur état de l'art pour plus de détails.

2.3 Multi-échelon

Dans cette section, nous nous intéressons aux principaux modèles où plusieurs étapes d'approvisionnement sont explicitement modélisées (voir figure 2.3). Le tableau 2.3 précise les hypothèses de chaque modèle présenté dans cette section.

Peu de travaux s'intéressent au contrôle des systèmes multi échelons avec les retours de produits. Dans ce cas, décrit figure 2.3, les différentes étapes de fabrications sont explicitées et les retours peuvent arriver à chacune de ces étapes.

Pionnier dans le domaine du multi échelon, Clark et Scarf (1960) étudient un système sans retour, de N stocks en série, à temps périodique, vente perdue, capacités d'approvisionnement infinies et coût linéaire. La demande est stochastique, et arrive uniquement sur l'étage aval. Avec ces hypothèses, ils prouvent que la politique base stock échelon est optimale. DeCroix et al. (2005) étendent ce résultat précédent à des demandes négatives (de la même façon que Fleischmann et Kuik, 2003 et Beltran et Krass, 2002). De plus ils proposent plusieurs méthodes de calcul pour déterminer les paramètres optimaux ou approchés de la politique optimale.

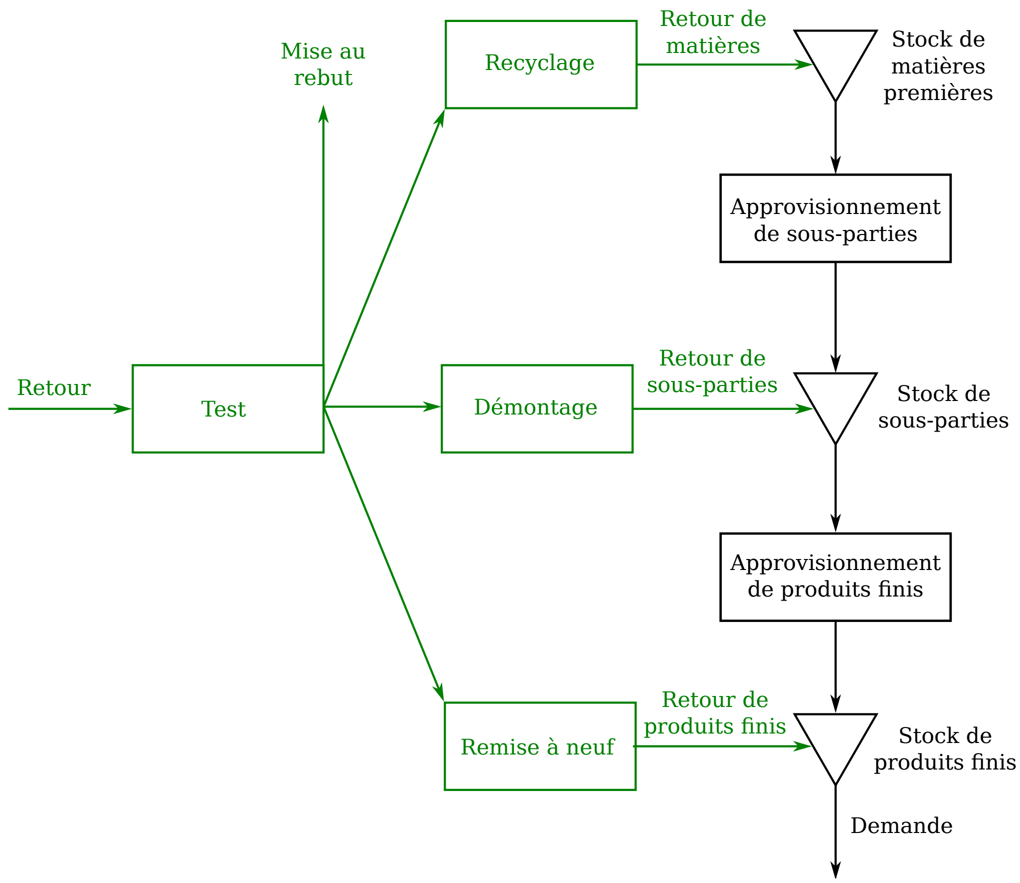


FIGURE 2.3 – Modèle multi-échelon avec des retours

Dans le cas d'un système d'assemblage, DeCroix et Zipkin (2005) montrent que si les retours arrivent à l'étage aval, alors le système est équivalent à un système en série et donc que les résultats de DeCroix et al. (2005) sur l'optimalité de la politique base stock échelon s'appliquent.

DeCroix (2006) étend le modèle de Clark et Scarf (1960), en ajoutant un flux de retour nécessitant une étape de démontage/réparation à l'étage amont. Avec une capacité infinie sur cette étape, il montre que la politique optimale est une politique base stock échelon. Le cas particulier avec un seul étage, se réduit au problème présenté par Simpson (1978).

Avec des coûts fixes, Mitra (2009) analyse un système avec deux échelons en série où les retours arrivent à l'étage amont après avoir été traités. L'auteur présente un modèle déterministe, ainsi qu'un modèle stochastique à temps continu et trouve les paramètres optimaux numériquement.

Aucun des modèles précédents ne représente la capacité d'approvisionnement. Avec un modèle de production et de stockage modélisé en file d'attente, Veatch et Wein (1992; 1994) considèrent un système à deux étages en série sans retour. La capacité de production est donc modélisée (par les files d'attentes) et les deux serveurs, produisant les produits un par un, ont des temps de production distribués selon des lois exponentielles. Les demandes sont

	Decroix et al. 2005	Decroix et Zipkin, 2005	Decroix 2006	Mitra 2009
Temps	Discret	Discret	Discret	Continu
Horizon	Infini	Infini	Fini	Infini
Critère de coût	Moyen	Actualisé	Actualisé	Moyen
Demande				
- Vente	Différée	Différée	Différée	Différée
- Distribution	Générale	Générale	Générale	Normale
Retour	À l'étage aval	Multiples	À l'étage amont	À l'étage amont
- Distribution	Générale	Générale	Générale	Normale
- Vs. Demande	Indépendant	Indépendant	Indépendant	Indépendant
- Rejet autorisé	Non	Non	Oui	Non
- Rejet post-accept.			Oui	
Approvisionnement	Série	Assemblage	Série	2 en série
- Capacité	Infinie	Infinie	Infinie	Infinie
- Coût fixe	Non	Non	Non	Oui
- Délai	Constant	Constant	Constant	Constant
Remise à neuf	Non	Non	Oui	Oui
- Capacité			Infinie	Infinie
- Coût fixe			Non	Oui
- Délai			Constant	Nul
- Stock			Oui	Oui
Politique	a : Base st. éch. b : Fixed buffer	Base st. éch.	Base st. éch.	(Complexe)
Optimalité	Oui (a)	Non	Oui	Non

TABLE 2.3 – Modèles multi-échelons avec des retours

Poissonniennes, les coûts linéaires et les demandes insatisfaites sont différées. Les auteurs montrent que la politique optimale n'est pas une politique base stock échelon et expliquent ce phénomène par la capacité limitée de production. De plus ils caractérisent la structure de la politique optimale. Enfin ils comparent numériquement plusieurs politiques heuristiques (Kanban, Conwip, Fixed buffer, Base stock échelon) avec la politique optimale. Ce modèle est généralisé avec des retours dans le chapitre 3.

2.4 Conclusion

La littérature sur les systèmes stochastiques avec des retours n'est pas nouvelle. Nous constatons que de nombreux travaux ont été réalisés pour les systèmes où la réutilisation des retours est directe. Cependant nous observons que la littérature sur les problèmes multidimensionnels (remise à neuf et multi-échelon) est plus limitée et que les politiques optimales ne sont déterminées que dans peu de cas.

Dans un premier temps, résumons l'état de l'art concernant les modèles stochastiques à réutilisation directe. Nous observons que beaucoup de travaux ont été réalisés avec et sans capacité d'approvisionnement. Dans ces travaux, l'optimalité des politiques considérées est généralement démontrée. Cependant, de nouveaux résultats analytiques semblent possibles avec une formulation en file d'attente. Ainsi les questions suivantes restent ouvertes. Quel est l'impact des ventes différées dans le cas où les clients annoncent à l'avance le renvoi potentiel de leur produit ? Dans ce même cas, quel est l'influence de la variabilité des temps de retours ? Enfin, quel est l'impact de l'avance d'information à la fois sur la demande et sur les retours ? Ces questions sont traitées dans le chapitre 5. Une autre piste de recherche non traitée dans ce document concerne l'étude d'un modèle à temps discret avec une capacité finie d'approvisionnement.

Concernant les modèles avec remise à neuf, nous constatons dans le tableau 2.2 que seul Simpson (1978) autorise le rejet des retours après acceptation. En pratique la politique optimale trouvée n'utilise jamais cette option. Une piste de recherche serait de considérer ce type de rejet avec des délais d'approvisionnement et de remise à neuf différents. De plus, Simpson (1978) démontre l'optimalité de la politique (S_m, s_d, S_r) pour un modèle avec une capacité de production infinie. Cette politique est-elle toujours optimale dans un modèle avec une capacité d'approvisionnement finie ? De la même façon, comment se comportent les nombreuses politiques heuristiques proposées, entre elles, et par rapport à la politique optimale ? Ces questions sont traitées dans le chapitre 4.

À notre connaissance aucun modèle multi-échelon avec des retours ne modélise de capacité d'approvisionnement finie. Il pourrait être intéressant de modéliser cette capacité et de quantifier son impact. De plus, la politique optimale d'un modèle où les retours arriveraient à plusieurs étages n'a jamais été caractérisée. Ces pistes de recherche sont étudiées dans le chapitre 3.

Chapter 3

Optimal control of a two-stage production-inventory system with product returns

We consider a two-stage production/inventory system with finite production capacities modelled by exponential servers. The downstream stage faces a Poisson demand. Each stage receives returns of products, according to independent Poisson processes, that can be used to serve demand. The problem is to control production to minimize discounted (or average) holding and backordering costs. For the single-stage problem, we fully characterize the optimal policy. We show that the optimal policy is base-stock and we derive an explicit formula for the optimal base-stock level. For the two-stage problem, we show that the optimal policy is characterized by state-dependent base-stock levels. In a numerical study, we investigate three heuristic policies for the two-stage problem: base-stock, Kanban and fixed buffer. The fixed-buffer policy obtains poor results while the relative performances of base-stock and Kanban policies depend on bottlenecks. We also show that returns have a non-monotonic effect on average costs and strongly affect the performances of heuristics. Finally, we observe that having returns at the upstream stage is preferable in some situations.

3.1 Introduction

The importance of product returns is growing in supply chains. Customers often can return products a short time after purchase, due to take-back commitments of the supplier. For instance, the proportion of returns is particularly important in electronic business where customers can not touch a product before purchasing it. Customers might also return used products a long time after purchase. This type of return has increased in recent years due to new regulations on waste reduction, especially in Europe. Some industries also encourage returns for economical and marketing reasons. Though different in nature, these two types of returns are similar from an inventory control point of view since they constitute a reverse flow which complicates decision making.

The inventory control literature on product returns is quite abundant (see e.g. Fleischmann et al. (1997); Ilgin and Gupta (2010); Zhou and Yu (2011)). However, most of the literature focusses on single-echelon systems with infinite production capacity. In this paper, we fill this gap by considering a two-stage production/inventory system with finite production capacity and product returns at each stage (see Figure 3.1). The flow of returns at the finished good (FG) inventory may result from remanufacturing, recycling, repairing or simply returning new products. The flow of returns at the work-in-process (WIP) inventory can also result from disassembly operations. For instance, the Kodak company reuses only some parts of cameras like circuit board, plastic body and lens aperture (Toktay et al., 2000). More precisely, we adopt a queueing framework to model production capac-

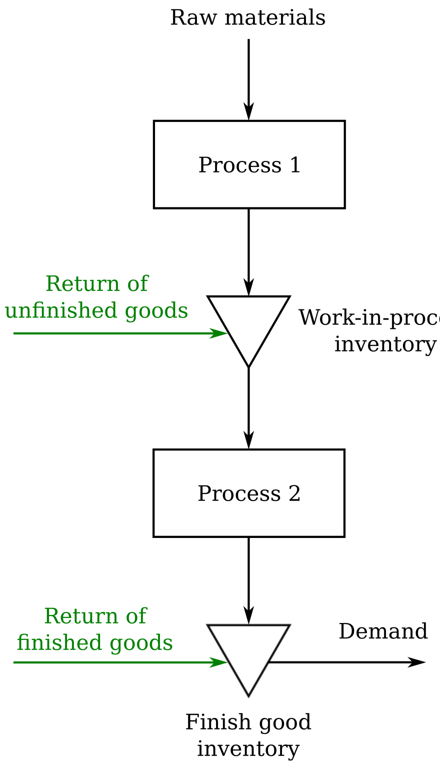


Figure 3.1: Two-stage production/inventory system with returns.

ity. Items are produced by servers one by one and each unit requires a random lead-time to be produced. We assume that each stage consists of a single exponential server and an output inventory. The downstream stage faces a Poisson demand. Each stage receives returns of products, according to independent Poisson processes, that can be used to serve demand. The problem is then to control production at each stage, in order to minimize discounted/average holding and backordering costs. We also study the single-stage problem which has not been studied in the literature. The single-stage problem is also helpful to analyze the two-stage problem in some situations. In what follows, we review the literature on single-echelon and multi-echelon systems with returns, before presenting in detail our contributions.

The literature on single-echelon systems is quite mature. Heyman (1977) considers an inventory system with independent Poisson demand and Poisson returns. Unsatisfied demands are backordered. Heyman assumes zero lead-times and linear costs for both manufacturing and remanufacturing. These strong assumptions imply that the optimal production policy is a make-to-order policy and that the optimal disposal policy is a simple threshold policy: when the inventory level exceeds a certain disposal threshold R, every returned item is disposed upon arrival. An explicit expression for the optimal disposal threshold is also derived. For a lost sale problem with exponential service times, Poisson demand and returns, Zerhouni et al. (2010) investigate the impact of ignoring dependency between demands and returns.

Fleischmann et al. (2002) consider a similar setting with deterministic manufacturing lead-time and fixed order cost. Again, remanufacturing lead-time and remanufacturing costs are neglected. They extend results standing for a system without returns by showing that the optimal policy is (s, Q) for the average-cost problem. For the periodic review version with a stochastic demand either positive or negative in each period, Fleischmann and Kuik (2003) show the average-cost optimality of an (s, S) policy. Simpson (1978) and Inderfurth (1997) consider a periodic-review problem where returns are held in a separate buffer until they are remanufactured or disposed of. When the remanufacturing lead-time is equal to the production leadtime and the costs are linear, they show that a three-parameter policy is optimal.

Apart from these optimal control papers, several heuristic policies have been investigated in the literature. van der Laan et al. (1996b) model the remanufacturing shop as an $M/M/c/(c + N)$ queue with c parallel servers and introduce the (s_p, Q_p, N) policy where s_p is the reorder point, Q_p the order quantity and any return is disposed whenever the number of products waiting for repair equals N. van der Laan et al. (1996a) extend this policy with the (s_p, Q_p, s_d, N) policy where returns are disposed when the stock level is above s_d. van der Laan and Salomon (1997) consider a model with correlated demand process and return process. They compare an (s_p, Q_p, Q_r, s_d) push-disposal policy with an $(s_p, Q_p, s_r, S_r, s_d)$ pull-disposal policy to coordinate manufacturing and remanufacturing decisions. For the push-disposal policy, returned products are remanufactured with batch size Q_r. For the pull-disposal policy, remanufacturing is initiated only when the finished good inventory is below s_r and the remanufacturable inventory is above S_r. Teunter and Vlachos (2002) complement the numerical study of the above model.

The literature on multi-echelon systems with returns is much more limited. In their seminal work (without returns), Clark and Scarf (1960) studies a series inventory system with N stages, periodic review, linear holding and backorder cost, no setup cost and stochastic demand at the downstream stage. They prove that a base-stock policy is optimal.

DeCroix et al. (2005) extend the results of Clark and Scarf (1960) to the case where demand can be negative. They also propose a method to compute a near optimal policy, explain how to extend their model when returns occur at different stages and compare the base-stock policies to fixed-buffer policies. DeCroix (2006) combines the multi-echelon structure of DeCroix et al. (2005) and the remanufacturing structure of Inderfurth (1997). DeCroix and Zipkin (2005) and Decroix et al. (2009) consider assemble-to-order systems with returns of components or finished product.

In production-inventory systems, replenishment is modelled in a different way than in pure inventory systems. Items are produced by servers one by one, or possibly by batches. Each unit, or batch, requires a random lead-time to be produced. Hence replenishments are capacitated in production-inventory systems. In line with this approach, Veatch and Wein (1994) consider a two-stage system with exponential server at each stage. Otherwise, their assumptions are similar to Clark and Scarf (1960). They prove that the optimal policy is never a base-stock policy. They investigate several classes of policies and compare them to the optimal policy. They conclude that the base-stock policy is generally the best heuristic. However, when the downstream station is the bottleneck, the Kanban policy is better. In another paper, Veatch and Wein (1992) show that the optimal policy is a state-dependent base-stock policy. Liberopoulos and Dallery (2003) investigates a generalized Kanban policy being a mix between Kanban and base-stock policy. In a deterministic environment, several papers have investigated capacitated production and/or remanufacturing (see e.g. Nahmias and Rivera (1979); Teunter (2001, 2004); Li et al. (2007)).

In this paper, we extend the model of Veatch and Wein (1994) by including Poisson returns at each stage. The structure of the optimal policy presented by Veatch and Wein (1992) pertains to the case with product returns. We show that the optimal policy is a complex state-dependent base-stock policy and we derive several monotonicity results for the base-stock levels. In several situations, we explain how the two-stage problem reduces to a simpler single-stage problem. Interestingly, the single-echelon problem has not been treated in the literature, when including Poisson returns. In this case, the optimal policy reduces to a simple base-stock policy and we are able to derive an explicit formula for the optimal base-stock level for both average-cost and discounted-cost problems. Such explicit formulas are very rare in inventory control theory, especially when returns are included. When service times, inter-arrival times and inter-return times are not exponential but have general i.i.d. distributions, we explain how to compute the optimal base-stock level by using results from the newsvendor problem.

When the two-stage problem does not reduce to a single-stage problem, the optimal policy has a complex form and might be difficult to implement in practice. To counter this, we evaluate the performances of three classes of heuristics (fixed buffer, base-stock and Kanban) which are reasonable with respect to the optimal policy structure. The fixed-buffer policy obtains poor results while the relative performances of base-stock and Kanban policies depend on bottlenecks, consistently with Veatch and Wein (1996). Moreover, we observe that return rates strongly affect the relative performances of heuristics.

Section 3.2 describes in detail the two-stage problem and presents two situations where the two-stage problem reduces to a single-stage problem. Section 3.3 provides a full characterization of the optimal policy for the single-stage system. Section 3.4 shows that the optimal policy for the two-stage system is a state-dependent base-stock policy. Section 3.5 investigates the performances of three heuristic policies for the two-stage problem. Finally, we conclude and discuss avenues for research in Section 3.6.

3.2 Assumptions and notations

We consider a two-stage production/inventory system in series which satisfies end-customer demand (see Figure 3.2). Station M_i, $i = 1, 2$, produces items one by one. The production lead-time of station M_i is exponentially distributed with rate μ_i. Preemption is allowed and works as follows. The processing of a job at station M_i can be interrupted at any point in time and continued latter. Because of the memoryless property of the exponential distribution, continuing a job is equivalent to restarting it from the beginning. Produced items are stocked in a buffer B_i just after M_i. The end buffer B_2 sees customer demands arriving according to a Poisson process with rate λ. We assume that backorders are allowed. At time t, the on-hand inventory at B_1 is denoted by $X_1(t)$ and the net on-hand inventory at B_2 is denoted by $X_2(t)$. When buffer B_1 is empty, M_2 can not start to produce.

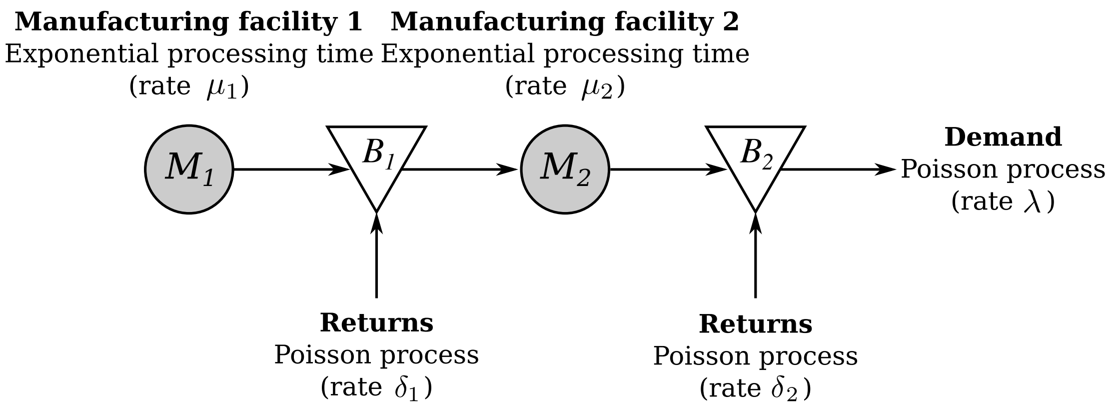


Figure 3.2: The two-stage $M/M/1$ make-to-stock queue with product returns.

Returns of products occur at buffer B_i according to an independent Poisson process with rate δ_i. When a return is accepted in buffer B_i, it can be used immediately as a new product (we neglect the remanufacturing lead-time). Another way to see these two return flows is to consider a situation where there is a single flow of returns for the whole system (rate $\delta_1 + \delta_2$) and, after an inspection, returned products are routed to the work-in-process inventory with probability $p_1 = \delta_1/(\delta_1 + \delta_2)$ and to the finished good inventory with probability $p_2 = \delta_2/(\delta_1 + \delta_2)$.

The system is stable if we have the following conditions on the parameters. First, the demand rate must be larger than the total return rate ($\lambda > \delta_1 + \delta_2$). Second, station M_2 must be able to process all returns at buffer B_1 ($\mu_2 > \delta_1$). Third, demands must be satisfied either by returns at buffer B_2 or by items produced by station M_2. Hence we have $\lambda < \delta_2 + \gamma_2$ where γ_2 is the maximum average production rate of station M_2. We have $\gamma_2 = \min[\mu_2, \mu_1 + \delta_1]$ since station M_2 can not produce when buffer B_1 is empty and buffer B_1 can be replenished at a maximum rate of $(\mu_1 + \delta_1)$. We can aggregate these three conditions in

$$\delta_1 + \delta_2 < \lambda < \min[\mu_1 + \delta_1, \mu_2] + \delta_2.$$

The system incurs in state $\mathbf{X}(t) = (X_1(t), X_2(t))$ a cost rate $c(X_1, X_2) = h_1 X_1 + h_2 X_2^+ + bX_2^-$ where h_i is the inventory holding cost per unit in stock per unit of time at buffer B_i, b is the backorder cost per unit of waiting demand per unit of time, $x^+ = \max[0, x]$ and $x^- = \max[0, -x]$. The unit return cost is c_i^r at stage i. As the optimal policy is independent of c_i^r, we set without loss of generality $c_i^r = 0$ for $i = 1, 2$.

A production policy π specifies when to produce for each stage. The discounted expected cost over an infinite horizon of a policy π, with initial state $\mathbf{x} = (x_1, x_2)$ and discount rate $\alpha > 0$, is

$$v_\alpha^\pi(\mathbf{x}) = E\left[\int_0^{+\infty} e^{-\alpha t} c(\mathbf{X}(t))\mathrm{d}t | \mathbf{X}(0) = \mathbf{x}\right].$$

Our objective is to find the optimal policy, denoted by $\pi^\star$, that minimizes the expected discounted cost $v_\alpha^\pi(\mathbf{x})$ over an infinite horizon. We denote by $v_\alpha^\star(\mathbf{x})$ the optimal value function:

$$v_\alpha^\star(\mathbf{x}) = \min_\pi v_\alpha^\pi(\mathbf{x}).$$

We are also interested in the average-cost problem

$$g^\star = \min_\pi \lim_{T\to\infty} \frac{E_\mathbf{x}^\pi\left[\int_0^T c(\mathbf{X}(t))\mathrm{d}t\right]}{T}.$$

There is a strong link between the discounted-cost problem and the average-cost problem. The average-cost optimal policy can be obtained as the limit of the discounted-cost optimal policy when α goes to zero. Moreover, the optimal average cost $g^\star$, is the limit of $\alpha v_\alpha^\star(\mathbf{x})$ when α goes to 0, for each $\mathbf{x}$. To justify these two properties, we use the results of Weber and Stidham (1987) which apply to problems with infinite state space and unbounded costs.

3.3 A full characterization of the optimal policy for the single-stage problem

Before considering the two-stage problem, we analyze the single-stage problem (see Figure 3.3), for which we are able to fully characterize the optimal policy. Veatch and Wein (1996) and Dusonchet and Hongler (2003) have investigated the single-stage problem. In this section, we extend their results to a single-stage problem including product returns. For the single-stage problem, we denote the system parameters by $\lambda, \mu, \delta, h, b, \alpha$ (demand

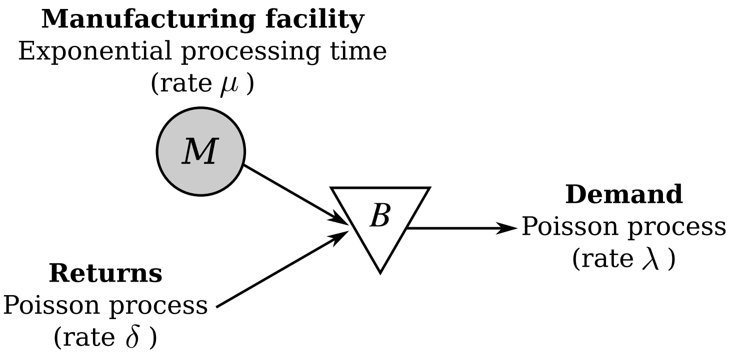


Figure 3.3: The single-stage $M/M/1$ make-to-stock queue with product returns.

rate, production rate, return rate, holding cost, backorder cost, discount rated). The net on-hand inventory is denoted by x and the cost rate is $c(x) = hx^+ + bx^-$. The problem is again to control production in order to minimize discounted or average costs.

3.3.1 Structure of the optimal policy

The problem of finding the optimal control policy can be formulated as a continuous-time Markov Decision Process (MDP). After uniformizing the MDP with rate $\tau = \lambda + \mu + \delta$, we can transform the continuous-time MDP into a discrete time MDP (Puterman, 1994). The optimal value function $v_\alpha^\star$ satisfies the following optimality equations:

$$v_\alpha^\star(x) = \mathcal{T} v_\alpha^\star(x), \ \forall x$$

with

$$\mathcal{T} v(x) = \frac{1}{\tau + \alpha}[c(x) + \mu \min[v(x), v(x+1)] + \lambda v(x-1) + \delta v(x+1)].$$

Theorem 3.3.1. *The optimal value function $v_\alpha^\star(x)$ is convex in x. The optimal policy for the discounted-cost problem (respectively the average-cost problem) is base-stock: there exists a base-stock level $S_\alpha^\star$ (respectively $S^\star$) such that it is optimal to produce if the stock level is smaller than $S_\alpha^\star$ (respectively $S^\star$) and to idle production otherwise.*

Proof. We first prove that operator $\mathcal{T}$ preserves convexity. Consider a convex value function v, i.e. such that $\Delta v(x) = v(x+1) - v(x)$ is non-decreasing in x. By assumption, the cost rate $c(x)$ is also convex. As mentioned by Koole (1998), the function $\min[v(x), v(x+1)]$ is also convex. The functions $v(x-1)$ and $v(x+1)$ are also convex. Finally $\mathcal{T}v$, as a non-negative linear combination of convex functions, is also convex.

As operator $\mathcal{T}$ is a contraction mapping, the fixed point theorem in a Banach space (Puterman, 1994) ensures that any sequence of value functions (v_n) defined as $v_{n+1} = \mathcal{T}v_n$ will converge to the optimal value function $v^\star_\alpha$, the unique solution of the optimality equations $v^\star_\alpha = \mathcal{T}v^\star_\alpha$.

If we take a null value function v_0, it is clear that v_0 is convex. By induction, we conclude that $v^\star_\alpha$ is convex. This property allows to define the threshold $S^\star_\alpha = \min[x : \Delta v^\star_\alpha(x) > 0]$ such that $\Delta v^\star_\alpha(x) \leq 0$ (produce) when $x < S^\star_\alpha$ and $\Delta v^\star_\alpha(x) > 0$ (idle) when $x \geq S^\star_\alpha$. For the average-cost problem, it suffices to use the property that the discounted-cost policy converges to the average-cost policy when α goes to 0 (Weber and Stidham, 1987). $\square$

3.3.2 Steady-state probabilities

In this subsection, we derive the steady state probabilities when the control policy is base-stock with a base-stock level S. In this case, the net on-hand inventory $X(t)$ evolves according to a continuous-time Markov chain (Figure 3.4). Define the ratios $\rho_1 = \frac{\lambda}{\mu+\delta}$ and

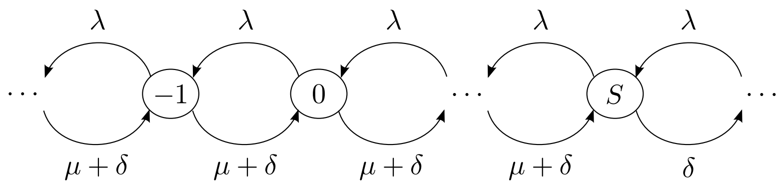


Figure 3.4: Markov chain in the backorder case with returns case.

$\rho_2 = \frac{\delta}{\lambda}$ where ρ_2 will be referred to as the return ratio. To ensure the stability of the number of backorders and the inventory level, we assume that $\rho_1 < 1$ and $\rho_2 < 1$. Let $p(i)$ be the steady-state probability to be in state i. We have

$$p(i) = \begin{cases} \rho_1^{S-i} p(S) & \text{if } i \leq S, \\ \rho_2^{i-S} p(S) & \text{if } i \geq S. \end{cases} \tag{3.1}$$

Using the normalization condition, $\sum_{i=-\infty}^{\infty} p(i) = 1$, we obtain

$$p(S) = \frac{(1-\rho_1)(1-\rho_2)}{1-\rho_1\rho_2}.$$

3.3.3 Average-cost problem

When computing the average cost, we must distinguish two cases: $S \geq 0$ and $S \leq 0$. These two cases are symmetrical (interchange h and b, ρ_1 and ρ_2 and replace S by $-S$). The average on-hand inventory $\bar{X}^+$ and the average number of backlogs $\bar{X}^-$ are given by

$$\bar{X}^+ = \sum_{i=0}^{+\infty} ip(i) = \begin{cases} \sum_{i=0}^{S} i\rho_1^{S-i}p(S) + \sum_{i=S+1}^{+\infty} i\rho_2^{i-S}p(S) & \text{if } S \geq 0 \\ \sum_{i=1}^{+\infty} i\rho_2^{i-S}p(S) & \text{if } S \leq 0 \end{cases}$$
$$= \begin{cases} S + p(S)\left[\frac{\rho_1}{(1-\rho_1)^2}\left(\rho_1^S - 1\right) + \frac{\rho_2}{(1-\rho_2)^2}\right] & \text{if } S \geq 0 \\ p(S)\frac{\rho_2^{-S+1}}{(1-\rho_2)^2} & \text{if } S \leq 0 \end{cases}$$

$$\bar{X}^- = -\sum_{i=-\infty}^{0} ip(i) = \begin{cases} -p(S)\sum_{i=-\infty}^{0} i\rho_1^{S-i} & \text{if } S \geq 0 \\ -p(S)\sum_{i=-\infty}^{S-1} i\rho_1^{S-i} - p(S)\sum_{i=S}^{0} i\rho_2^{i-S} & \text{if } S \leq 0 \end{cases}$$
$$= \begin{cases} p(S)\frac{\rho_1^{S+1}}{(1-\rho_1)^2} & \text{if } S \geq 0 \\ -S + p(S)\left[\frac{\rho_2}{(1-\rho_2)^2}\left(\rho_2^{-S} - 1\right) + \frac{\rho_1}{(1-\rho_1)^2}\right] & \text{if } S \leq 0 \end{cases}$$

After some algebraic operations, the average cost $g(S) = h\bar{X}^+ + b\bar{X}^-$ can be expressed as

$$g(S) = \begin{cases} h\left\{S + p(S)\left[\frac{\rho_1}{(1-\rho_1)^2}\left(-1 + \frac{h+b}{h}\rho_1^S\right) + \frac{\rho_2}{(1-\rho_2)^2}\right]\right\} & \text{if } S \geq 0, \\ b\left\{-S + p(S)\left[\frac{\rho_2}{(1-\rho_2)^2}\left(-1 + \frac{b+h}{b}\rho_2^{-S}\right) + \frac{\rho_1}{(1-\rho_1)^2}\right]\right\} & \text{if } S \leq 0. \end{cases} \tag{3.2}$$

and

$$\begin{cases} g(S+1) - g(S) = \frac{(1-\rho_2)[h-(h+b)\rho_1^{S+1}]+h(1-\rho_1)\rho_2}{1-\rho_1\rho_2} & \text{if } S \geq 0, \\ g(S-1) - g(S) = -\frac{(1-\rho_1)[b-(b+h)\rho_2^{-S+1}]+b(1-\rho_2)\rho_1}{1-\rho_2\rho_1} & \text{if } S \leq 0. \end{cases}$$

The quantity $\Delta g(S) = g(S+1) - g(S)$ is increasing in S, which implies that $g(\cdot)$ is convex. Hence the average cost is minimized for $S^\star = \min[S : \Delta g(S) > 0]$. This property implies the following theorem.

Theorem 3.3.2. *The average-cost optimal base-stock level $S^\star$ is*

$$S^* = \begin{cases} \left\lfloor \frac{\ln\left(\frac{1-\rho_1\rho_2}{1-\rho_2}\frac{h}{h+b}\right)}{\ln \rho_1} \right\rfloor \geq 0 & \textit{if} \quad \frac{1-\rho_1\rho_2}{1-\rho_2}\frac{h}{h+b} \leq 1, \\ \left\lceil -\frac{\ln\left(\frac{1-\rho_2\rho_1}{1-\rho_1}\frac{b}{b+h}\right)}{\ln \rho_2} \right\rceil \leq 0 & \textit{else.} \end{cases}$$

Based on Theorem 3.3.2, we can easily establish several properties of the optimal base-stock level. First, $S^\star$ is a decreasing function of the return rate δ. When the return rate is increasing, it is better off diminishing the base-stock level in order to limit excess inventory. When $\delta = 0$, we re-obtain the result obtained by Veatch and Wein (1996) in a system without returns:

$$S^\star = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \frac{\lambda}{\mu}} \right\rfloor \text{ if } \delta = 0.$$

When δ goes to λ, ρ_2 goes to 1 and $S^\star$ goes to infinity. In presence of returns, the base-stock level can take any negative integer value. Without returns, the optimal base-stock level is always non-negative.

3.3.4 Discounted-cost problem

It is more complex to compute analytically the optimal base-stock level in the discounted cost case. Denote by $v_\alpha^S(x)$ the expected discounted cost when the base-stock level is S, the initial inventory level is x and the discount rate is α. The following lemma establishes an explicit formula for the discounted cost.

Lemme 3.3.3.

$$v_\alpha^S(S) = \begin{cases} \frac{h}{\alpha}\left(S + \alpha B\left[\frac{\beta_1}{(1-\beta_1)^2}\left(-1 + \frac{h+b}{h}\beta_1^S\right) + \frac{\beta_2}{(1-\beta_2)^2}\right]\right) & \textit{if } S \geq 0 \\ \frac{b}{\alpha}\left(-S + \alpha B\left[\frac{\beta_2}{(1-\beta_2)^2}\left(-1 + \frac{b+h}{b}\beta_2^{-S}\right) + \frac{\beta_1}{(1-\beta_1)^2}\right]\right) & \textit{if } S \leq 0 \end{cases} \tag{3.3}$$

where

$$B = \frac{1}{\alpha}\frac{(1-\beta_1)(1-\beta_2)}{1-\beta_1\beta_2},$$
$$\beta_1 = \frac{\alpha + \lambda + \delta + \mu - \sqrt{(\alpha+\lambda+\mu+\delta)^2 - 4\lambda(\mu+\delta)}}{2(\mu+\delta)},$$
$$\beta_2 = \frac{\alpha + \lambda + \delta - \sqrt{(\alpha+\lambda+\delta)^2 - 4\lambda\delta}}{2\lambda}.$$

The proof of this lemma is provided in appendix.

When α goes to 0, β_1 goes to ρ_1, β_2 goes to ρ_2 and αB goes to $p(S)$. Therefore $\alpha v_\alpha^S(S)$ goes to the average cost $g(S)$, given in Equation (3.2), consistently with Weber and Stidham (1987).

We have $v_\alpha^\star(S_\alpha^\star) = \min_S v_\alpha^S(S)$. Similarly to the average-cost problem, we have

$$\begin{cases} v_\alpha^S(S+1) - v_\alpha^S(S) = \frac{1}{\alpha}\frac{(1-\beta_2)[h-(h+b)\beta_1^{S+1}]+h(1-\beta_1)\beta_2}{1-\beta_1\beta_2} & \text{if } S \geq 0 \\ v_\alpha^S(S-1) - v_\alpha^S(S) = -\frac{1}{\alpha}\frac{(1-\beta_1)[b-(b+h)\beta_2^{-S+1}]+b(1-\beta_2)\beta_1}{1-\beta_2\beta_1} & \text{if } S \leq 0 \end{cases}$$

The quantity $\Delta v_\alpha^S(S) = v_\alpha^S(S+1) - v_\alpha^S(S)$ is increasing in S and again $S_\alpha^\star = \min[S : \Delta_S v_\alpha^S(S) > 0]$.

Theorem 3.3.4. *The optimal base-stock level $S_\alpha^\star$ of the discounted problem is*

$$S_\alpha^\star = \begin{cases} \left\lfloor \dfrac{\ln\left(\frac{1-\beta_1\beta_2}{1-\beta_2}\frac{h}{h+b}\right)}{\ln \beta_1} \right\rfloor \geq 0 & \text{if} \quad \frac{1-\beta_1\beta_2}{1-\beta_2}\frac{h}{h+b} \leq 1, \\ \left\lceil -\dfrac{\ln\left(\frac{1-\beta_2\beta_1}{1-\beta_1}\frac{b}{b+h}\right)}{\ln \beta_2} \right\rceil \leq 0 & \text{else.} \end{cases}$$

Theorem 3.3.4 is consistent with Theorem 3.3.2: When α goes to 0, $S_\alpha^\star$ goes to $S^\star$. Theorem 3.3.4 is also consistent with the results of Dusonchet and Hongler (2003) who consider the case without returns ($\delta = 0$).

3.3.5 General distributions

In this subsection only, we relax the assumption of exponential distributions and simply assume that service times, inter-arrival times and inter-return times are identically and independently distributed. In this case, the optimal policy can be very complicated and we focus on the class of base-stock policies. Consider a base-stock policy with base-stock level S. Let $X(t)$ be the net on-hand inventory level at time t and define $N(t) = S - X(t)$. The probability distribution of $N(t)$ is independent of S. Denote by $p(i)$ the steady-state probability of $N(t)$ and by $F(i) = \sum_{x=-\infty}^{i} p(x)$ the cumulative distribution function. The average cost is then

$$\begin{aligned} g(S) &= hE(X^+) + bE(X^-) = hE(S-N)^+ + bE(S-N)^- \\ &= h\sum_{i=0}^{S}(S-i)p(i) + b\sum_{i=S}^{\infty}(i-S)p(i). \end{aligned}$$

We recognize the objective function of a newsboy problem where the order quantity is S, the stochastic demand is N, the shortage cost is b and the holding cost is h. The optimal order quantity for the newsboy model is

$$S^\star = \min\left[S : F(S) > \frac{b}{h+b}\right]. \tag{3.4}$$

In the special case of an $M/M/1$ make-to-stock queue with Poisson returns, (3.4) yields to Theorem 3.3.2. For other distributions, numerical methods or simulation can be used to compute $F(\cdot)$.

When considering a lost-sale version of our problem, it can be shown that the optimal policy is base-stock. However, it is not possible to derive closed-form expressions for the optimal base-stock level.

3.4 A partial characterization of the optimal policy for the two-stage problem

The two-stage problem is more complex to analyze and it seems intractable to fully characterize the optimal policy. In this section, we provide some characteristics of the optimal policy, before investigating the performances of several heuristics, in the next section.

Again, the two-stage problem can can be formulated as a continuous-time Markov Decision Process (MDP). After uniformizing the MDP with rate $\tau = \lambda + \mu_1 + \mu_2 + \delta_1 + \delta_2$, we can transform the continuous-time MDP into a discrete time MDP (Puterman, 1994). The optimal value function $v^\star_\alpha$ satisfies the following optimality equations:

$$v^\star_\alpha(\mathbf{x}) = \mathcal{T} v^\star_\alpha(\mathbf{x}),\ \forall \mathbf{x}$$

with

$$\begin{aligned}
\mathcal{T}v(\mathbf{x}) &= \tfrac{1}{\tau+\alpha}\left[c(\mathbf{x}) + \lambda v(\mathbf{x}-\mathbf{e}_2) + \textstyle\sum_{i=1}^{2}(\delta_i v(\mathbf{x}+\mathbf{e}_i) + \mu_i T_i v(\mathbf{x}))\right],\\
T_1 v(\mathbf{x}) &= \min[v(\mathbf{x}), v(\mathbf{x}+\mathbf{e}_1)],\\
T_2 v(\mathbf{x}) &= \begin{cases} \min[v(\mathbf{x}), v(\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2)] & \text{if } x_1 > 0,\\ v(\mathbf{x}) & \text{else.}\end{cases}
\end{aligned}$$

In the optimality equations, $\mathbf{e}_1$ and $\mathbf{e}_2$ stand for vectors $(1,0)$ and $(0,1)$. In order to derive structural properties of the optimal policy, we will show that the optimal value function belongs to the following set of value functions V.

Definition 3.4.1. A value function v belongs to V if for all $\mathbf{x}$:

(a) $v(\mathbf{x}+\mathbf{e}_1) - v(\mathbf{x}) \le v(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2) - v(\mathbf{x}+\mathbf{e}_2)$,

(b) (i) $v(\mathbf{x}+\mathbf{e}_2) - v(\mathbf{x}+\mathbf{e}_1) \ge v(\mathbf{x}+\mathbf{e}_2+\mathbf{e}_1) - v(\mathbf{x}+2\mathbf{e}_1)$,
and
(ii) $v(\mathbf{x}+\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_2) \ge v(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2) - v(\mathbf{x}+2\mathbf{e}_2)$,

(c) $v(\mathbf{x}+\mathbf{e}_1) - v(\mathbf{x}) \le v(\mathbf{x}+2\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_1)$,

(d) $v(\mathbf{x}+\mathbf{e}_2) - v(\mathbf{x}) \le v(\mathbf{x}+2\mathbf{e}_2) - v(\mathbf{x}+\mathbf{e}_2)$.

In Koole (1998), property (a) is called supermodularity and denoted by $Super(1,2)$. Property (b) is called superconvexity and denoted by $SuperC(1,2)$. Finally (c) and (d) refer to the convexity of v in x_1 and x_2 and are denoted by $Conv(1)$ and $Conv(2)$. Properties (a) and (b) imply together properties (c) and (d).

The following theorem shows that the optimal value function satisfies all these properties and consequently provides a characterization of the optimal policy.

Theorem 3.4.1. *The optimal value function $v^\star_\alpha$ belongs to V and the discounted-cost optimal policy is a state-dependent base-stock policy. There exists two switching curves $S^\star_1(x_2)$ and $S^\star_2(x_1)$ such that*

- *Produce at M_1 if and only if $x_1 < S_1^\star(x_2)$. Moreover $S_1^\star(x_2)-1 \leq S_1^\star(x_2+1) \leq S_1^\star(x_2)$.*

- *Produce at M_2 if and only if $x_2 < S_2^\star(x_1)$. Moreover $S_2^\star(x_1) \leq S_2^\star(x_1+1)$.*

Proof. The proof is again by induction (see proof of Theorem 3.3.1). The operator $\mathcal{T}$ is a convex combination of operators that propagate $Super(1,2)$ and $SuperC(1,2)$ (Koole, 1998). As a result, if a value function v is in V, then the value function $\mathcal{T}v$ is also in V. By induction, we conclude that $v_\alpha^\star \in V$.

As $v_\alpha^\star \in V$, we can define the thresholds $S_1^\star(x_2)$ and $S_2^\star(x_1)$. The threshold $S_1^\star(x_2) = \min[x_1|v(x_1+1,x_2) - v(x_1,x_2) > 0]$ is well defined since $v_\alpha^\star$ is $Conv(1)$. The threshold $S_2^\star(x_1) = \min[x_2|v(x_1-1,x_2+1) - v(x_1,x_2) > 0]$ is also well defined since $v_\alpha^\star$ is $SuperC(1,2)$,

The monotonicity results on the switching curves are also implied by the fact that $v_\alpha^\star \in V$. For instance, $Super(1,2)$ ensures that $S_1^\star(x_2+1) \leq S_1^\star(x_2)$ and $SuperC(1,2)$ ensures that $S_1^\star(x_2) - 1 \geq S_1^\star(x_2+1)$. The other monotonicity results are obtained in a similar way. □

The structure of the optimal policy pertains to the average-cost problem as explained at the end of Section 3.2. With lost sales instead of backorders, it can be shown similarly that the optimal policy has the same structure. Figure 3.5 illustrates Theorem 3.4.1 on a numerical example. The computational procedure to obtain this curve is explained in appendix.

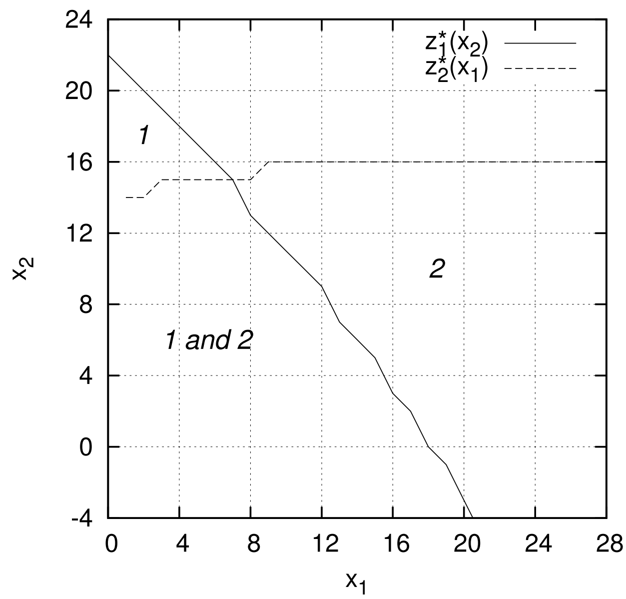


Figure 3.5: An illustration of the average cost optimal policy ($\mu_1 = 0.5$, $\mu_2 = 0.8$, $\delta_1 = 0.3$, $\delta_2 = 0.3$, $\lambda = 1$, $h_1 = 1$, $h_2 = 2$, $b = 4$). Number i means that station M_i produces in this region.

In several situations, the two-stage problem reduces to a single-stage problem (see Figure 3.3). A first situation where the two-stage problem reduces to the single stage problem is when the holding cost h_1 at the first stage is very low. This typically occurs when the value of the raw material is negligible with respect to the value of the finished good. A second situation is when the second station is much faster than the first station (μ_2 very high). More details on these reductions are provided in the appendix.

3.5 Heuristic policies for the two-stage problem

In this section, we investigate the performances of three simple and classical policies: the fixed-buffer policy, the base-stock policy and the Kanban policy. Each heuristic can be described by two parameters S_1 and S_2. The production control of each class of policies is detailed in Table 3.1.

In each class of policies, we compute the optimal average-cost policy parameter values (details on the computational procedure are given at the end of the appendix) for all combinations of the following values:

$$\lambda = \{1\}, \mu_1 = \{1, 1.5, 2\}, \mu_2 = \{1, 1.5, 2\}, \delta_1 = \{0, 0.3, 0.6, 0.8\},$$
$$\delta_2 = \{0, 0.3, 0.6, 0.8\}, h_1 = \{1\}, h_2 = \{0.5, 1, 10\}, b = \{0.5, 1, 10, 100\}.$$

If we restrict to systems satisfying the stability conditions (3.2), we obtain results for 912 instances summarized in Table 3.2. In this table, Δg is the % cost increase for using a heuristic policy (with parameters set optimally) instead of the optimal policy.

We observe that the base-stock policy is generally the best heuristic and outperforms other policies in 68% of cases, with a Δg less than 10% in 88% of cases. The fixed-buffer policy is the worst by far and is outperformed by other policies in 99.4% of cases. The Kanban policy is the best heuristic in 31.4%.

When station M_1 is the bottleneck ($\mu_2/(\mu_1 + \delta_1) \ll 1$), the base-stock policy generally performs better than the Kanban policy. It is the reverse when station M_2 is the bottleneck. These results are consistent with Veatch and Wein (1994) who treat the problem without returns. In Figure 3.6, we consider the influence of δ_1 on the performance of base-stock and Kanban policies. We have chosen an instance such that Station 1 is the bottleneck when δ_1 is small and Station 2 is the bottleneck when δ_1 is large. Expectedly, we observe that the base-stock policy performs better when Station 1 is the bottleneck and the Kanban policy performs better when Station 2 is the bottleneck.

In Figure 3.7, we observe that return rates have a non-monotonic effect on average costs. To explain this behavior, let's rewrite the stability condition (3.2) as

$$\lambda - \mu_1 - \delta_2 < \delta_1 < \lambda - \delta_2,$$
$$\lambda - \mu_2 < \delta_2 < \lambda - \delta_1.$$

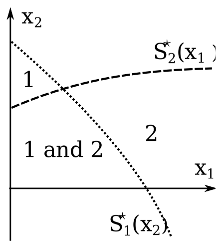
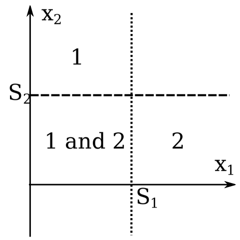
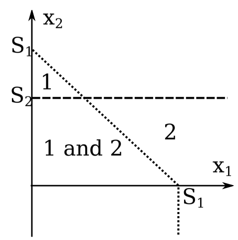
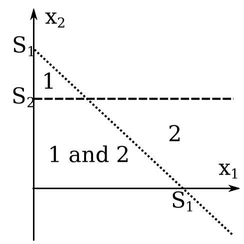
Policies	Optimal policy ($\pi^\star$)	Fixed buffer (FB)	Kanban (KB)	Base stock echelon (BSE)
Produce in M_1 if	$x_1 + x_2 < S_1$	$x_1 < S_1$	$x_1 + x_2^+ < S_1$	$x_1 + x_2 < S_1$
Produce in M_2 if	$x_2 < S_2$	$x_2 < S_2$	$x_2 < S_2$	$x_2 < S_2$
Switching curves				

Table 3.1: Production control policies.

	Fixed-buffer	Base-stock	Kanban
% of instances where the heuristic is the best	0.6	68.0	31.4
Average Δg (%)	29.0	3.8	9.8
Minimum Δg (%)	0.60	0.0	0.0
Maximum Δg (%)	590	51.4	150
% of instances with $\Delta g \in [0\%; 1\%[$	0.4	45.6	25.4
% of instances with $\Delta g \in [1\%; 5\%[$	14.2	31.4	26.6
% of instances with $\Delta g \in [5\%; 10\%[$	5.6	11.2	13.7
% of instances with $\Delta g > 5\%$	79.8	11.8	34.3

Table 3.2: Quantitative comparison of heuristics.

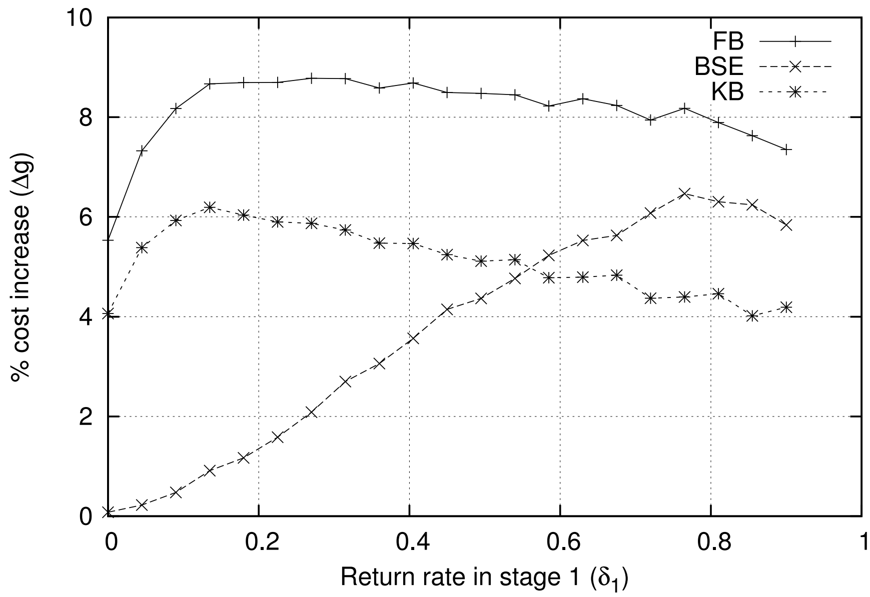


Figure 3.6: Influence of returns on the performances of heuristics ($\mu_1 = 1.1, \mu_2 = 1.2, \lambda = 1, \delta_2 = 0, h_1 = 1, h_2 = 5, b = 4$).

When δ_1 (resp. δ_2) goes to $\lambda - \delta_2$ (resp. to $\lambda - \delta_1$), the total average on-hand inventory goes to infinity, so does the average cost. On the other hand, when δ_1 (respectively δ_2) decreases to $\lambda - \mu_1 - \delta_2$ (respectively to $\lambda - \mu_2$), the average number of backorders goes to infinity, so does the average cost. These phenomenons do not appear when production is uncapacitated (Fleischmann et al., 2002).

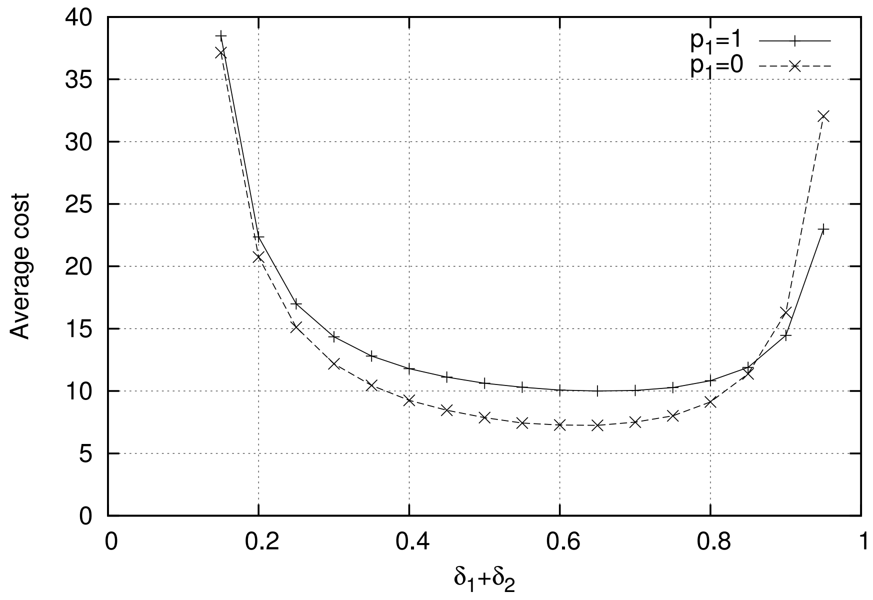


Figure 3.7: Optimal average cost in function of type and quantity of returns ($\mu_1 = 0.9$, $\mu_2 = 1.5$, $\lambda = 1$, $h_1 = 1$, $h_2 = 2$, $b = 4$), with $p_1 = \delta_1/(\delta_1 + \delta_2)$.

On the same figure, we observe that when the return rate is high ($\delta_1 + \delta_2 \geq 0.9$), it is preferable to return products in the first stage. In this case, returned products have to stay a long time in the system before being consumed by demand. So the system prefers to keep returns in the queue with the lowest holding cost ($h_1 < h_2$). When the return rate is smaller ($\delta_1 + \delta_2 \leq 0.9$), it is preferable to have returns at stage 2 in order to satisfy the demand quickly.

3.6 Conclusions and future research

In this paper, we consider a two-stage production-inventory system with returns. Unlike most of the literature on inventory control with returns, we assume that production is capacitated. To model production capacity, we adopt a queuing framework.

Interestingly, the single-echelon make-to-stock queue problem has not been treated in the literature, when including Poisson returns. In this case, the optimal policy reduces to a simple base-stock policy and we are able to derive an explicit formula for the optimal base-stock level for both average-cost and discounted-cost problems.

For the two-stage problem, we show that the optimal policy is characterized by two switching curves with several monotonicity properties. Based on this characterization, we investigate the performances of three heuristics. The fixed-buffer policy obtains poor results while the relative performances of base-stock and Kanban policies depend on bottlenecks. We also show that returns have a non-monotonic effect on average costs and strongly affect the performances of heuristics. Finally, we observe that having returns at the upstream stage is preferable in some situations.

In this paper, we have assumed that returns where always accepted in the system. A first avenue for research is to control arrivals of returns. A return can be either accepted with an acceptance cost or rejected with a rejection cost. For the single-stage problem, the optimal policy is likely to be an (R, S) policy stating to accept returns when the inventory level is below R and to produce when the inventory level is below S. For the two-stage problem, the optimal policy should be characterized by two production/idle switching curves and two accept/reject switching curves. Another avenue for research is to model explicitly the remanufacturing process. In this case, returned products are first kept in a remanufacturable inventory, before being remanufactured.

Chapter 4

Coordination of manufacturing, remanufacturing and return acceptance in a hybrid production-inventory system

This chapter deals with the coordination of manufacturing, remanufacturing and returns acceptance in a hybrid production-inventory system. We use a queuing control framework, where manufacturing and remanufacturing are modeled by single servers with exponentially distributed processing times. Customer demands and returned products arrive in the system according to independent Poisson processes. A returned product can be either rejected or accepted. When accepted, a return is placed in a remanufacturable product inventory. New products and remanufactured products are placed in a serviceable product inventory and customer demand can be satisfied as well by new or remanufactured products. The following costs are included: stock keeping, backorder, manufacturing, remanufacturing, acceptance and rejection costs. We show that the optimal policy is characterized by two state-dependent base-stock thresholds for manufacturing and remanufacturing and one state-dependent acceptance threshold. We obtain monotonicity results for the corresponding switching curves. We compare several types of controls on manufacturing, remanufacturing and returns acceptance via a numerical study. We present a new heuristic policy and we adapt relevant heuristic policies from the literature to compare their performances with the performance of the optimal policy. Insights about efficient controls and topics for further research are indicated.

4.1 Introduction

During the last two decades, quite some attention has been paid to the problem of jointly controlling the manufacturing of new products and remanufacturing of returned products, see e.g. Ilgin and Gupta (2010) or Rubio et al. (2008).

In addition to the joint control of manufacturing and remanufacturing, another important issue is whether or not to accept returns. There are many situations in practice where controlling the acceptance of returns can result in considerable cost savings, typically when the costs related to accepting a return are high. These costs include, among others, transportation costs (related to the collection of returns), stock keeping costs, recovery costs, disposal costs are high.

In this chapter, we characterize the optimal control of manufacturing, remanufacturing and returns acceptance for the situation shown in Figure 4.1.

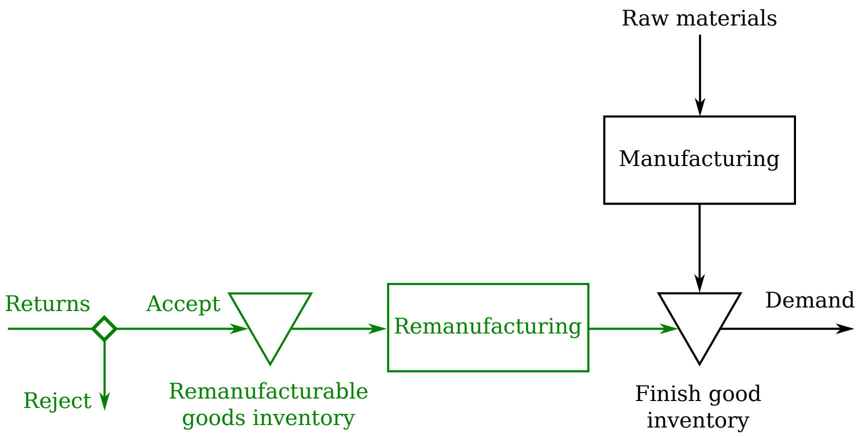


Figure 4.1: Hybrid system with manufacturing and remanufacturing.

The setup of the rest of the chapter is as follows. Section 4.2 provides a literature review and points out our contributions to literature and practice. In Section 4.3 the assumptions that we make are presented. The structure of the optimal policy is derived in Section 4.4. The special cases where returns can be reused without remanufacturing is presented in Section 4.5. In Section 4.6 we investigate several heuristic control policies for manufacturing, remanufacturing and return acceptance. We also compare numerically these heuristic policies to the optimal policy. The paper ends with a brief summary of the main results and some directions for further research.

4.2 Literature review

Manufacturing, remanufacturing as well as return acceptance decisions can be based on different data. Two important data in this context are the serviceable product inventory position I and the remanufacturable product inventory R. More precisely, I denotes the number of products in the serviceable stock plus the products actually being manufactured and remanufactured minus backlogs and R denotes the number of products in the stock of accepted returned products not yet remanufactured. Hereafter we discuss the control policies presented in the literature, where we explicitly indicate which type of data is used in these policies.

Simpson (1978) studies a pure inventory system with periodic review. Demands and returns are i.i.d. through the time horizon. Returns can be accepted or rejected at the end of each period. Unsatisfied demand is backlogged. The author assumes constant equal lead times for manufacturing and remanufacturing. He proves the optimality of a (S_m, S_r, S_a) policy, with S_m the manufacture up to level parameter, S_r the remanufacture up to level parameter, and S_a the accept (returns) up to level parameter. Here the manufacturing decision and the acceptance decision are based on $I+R$ while the remanufacturing decision is based on I. To explicitly show the link between decisions and the data used for them, we denote the policy by $(S_m[I+R], S_r[I], S_a[I+R])$.

Kiesmüller (2003) studies the same model with non-equal lead times. All returns have to be accepted. She proposes two heuristic policies: $(S_m[I], S_r[I])$ and $(S_m[I+R], S_r[I])$. Note that she defines a modified inventory position I and a modified remanufacturable inventory R, which take into account not the whole products actually being manufactured and remanufactured but only some part. For instance when the manufacturing lead time is larger than the remanufacturing lead time she proposes to consider only older products actually being manufactured in the inventory position I instead of all the products actually being manufactured.

Van der Laan and Teunter (2006) consider a continuous review model. Demand and returns occur according to independent Poisson processes. There is a set-up cost for manufacturing and remanufacturing. Manufacturing and remanufacturing lead times are equal. A $(s_m[I], Q_m, Q_r)$ policy is proposed, with s_m the reorder point where the system starts to manufacture Q_m products. The control of remanufacturing is a push policy by batches: as soon as there are Q_r products in the remanufacturable stock, these products are sent to remanufacturing. The authors compare this policy with a pull remanufacturing policy $(s_m[I], Q_m, s_r[I], Q_r)$, with s_r the reorder point for remanufacturing products. The authors give approximate formulas for the optimal values of the different parameters and compare them to the optimal parameter values in a numerical study.

Van der Laan et al. (1996b) study a model with the possibility of rejecting returns upon arrival. The manufacturing lead time is constant. There are a finite number of remanufacturing servers with exponentially distributed remanufacturing times. There is a setup cost for manufacturing and no setup cost for remanufacturing. The authors propose a $(s_m[I+R], Q_m, S_a[R])$ push remanufacturing policy and derive an analytical expressions for the average cost.

Van der Laan et al. (1996a) generalize the above policy via a $(s_m[I+R], Q_m, S_a^1[I+R]$, $S_a^2[R])$ push remanufacturing policy. For a system with remanufacturable stock holding cost, returns are accepted if both $I+R < S_a^1$, and $R < S_a^2$ hold. This entrance policy resembles the Kanban generalized policy proposed by Liberopoulos and Dallery (2003).

Van der Laan and Salomon (1997) consider a model with demand and return inter-occurrence times being Coxian-2 distributed. The demand and return process are correlated. The authors compare the $(s_m[I], Q_m, Q_r, S_a[I])$ push remanufacturing policy with the $(s_m[I], Q_m, s_r[I], \bar{S}_r, S_a[R])$ pull remanufacturing policy, where the system remanufactures $\bar{S}_r - I$ products if $I \leq s_r$. Teunter and Vlachos (2002) complement the numerical study of the above model.

Gupta and Korugan (2000) briefly examine a Kanban policy for a similar model where unsatisfied demand is backlogged. For the one stage remanufacturing problem the structure of the Kanban policy is $(S_m[I], S_r[I], S_a[I' + R])$ with I' the inventory position without taking into account backlogged demand.

The main contributions of this chapter are (1) as far as we know we characterize for the first time the structure of the optimal policy for a situation with stochastic manufacturing and remanufacturing times, capacity restrictions and returns acceptance/rejection with autonomous supply of returns and uncertain demand, and (2) a comparison of the performance of related existing heuristics with the performance of the optimal policy for the situation studied in this chapter via a numerical study.

4.3 Problem formulation

We consider a single item production-inventory hybrid system where product demand can be fulfilled both by manufacturing of new products and remanufacturing of products that are returned to the company (see Figure 4.2).

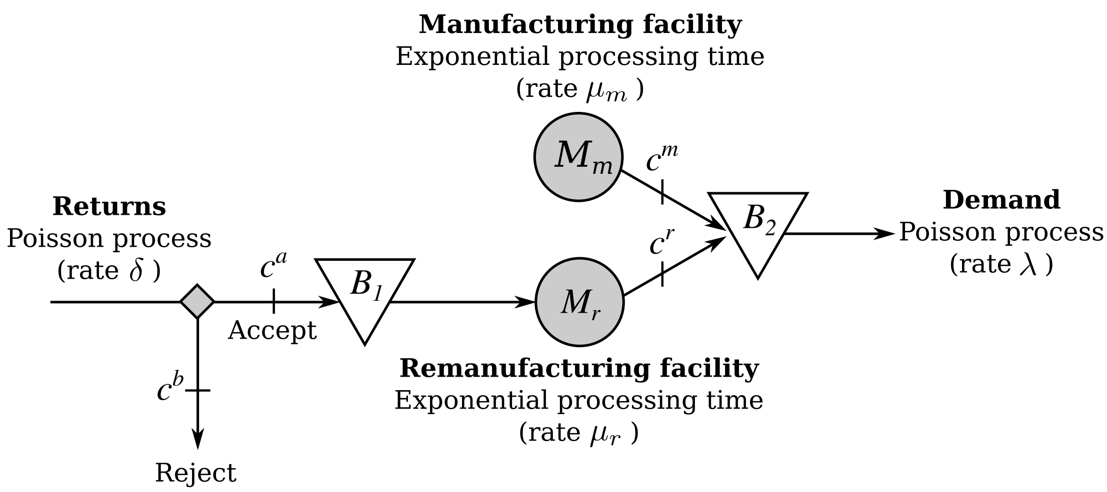


Figure 4.2: Model of the hybrid system with manufacturing and remanufacturing.

The manufacturing server (M_m) produces and the remanufacturing server (M_r) processes products one by one with exponentially distributed processing time (rates μ_m and μ_r). These two servers can start or stop processing at any time (preemption is allowed). The manufacturing operation creates new serviceable products to fill buffer B_2. Returns enter according to a Poisson process with rate δ independent of the demand process. They can be either rejected upon arrival with cost c^b or accepted with cost c^a and placed into buffer B_1. When a return is accepted, it has to be remanufactured before it can be used as a serviceable product. It is not allowed to dispose a return once accepted. We assume that all accepted returns can be remanufactured. Via remanufacturing, remanufacturable products move from buffer B_1 to buffer B_2. Buffer B_2 sees customer demands arriving according to a Poisson process with rate λ. We assume that backorders are allowed and there are no storage restrictions. At time t, the net inventory level at B_2 (resp. B_1) is denoted by $X_2(t)$ (resp. $X_1(t)$). Note that $X_2(t)$ can be negative due to backorders. When buffer B_1 is empty, remanufacturing is not possible. Let c^r be the unit remanufacturing cost and c^m the unit manufacturing cost. Per unit of time, the system incurs in state $\mathbf{x} = (x_1, x_2)$ a cost rate $C(\mathbf{x}) = h_1 x_1 + h_2 x_2^+ + b x_2^-$ where h_i is the unit inventory holding cost in buffer B_i, b is the unit backlog cost, $x^+ = \max\{0, x\}$ and $x^- = -\min\{0, x\}$. To ensure the stability of the system, we have to assume that the demand rate is smaller than the total production capacity rate: $\lambda < \mu_m + \min\{\mu_r, \delta\}$.

A production policy $\boldsymbol{\pi}$ specifies when to manufacture and remanufacture products and when accept the returns. Every policy $\boldsymbol{\pi}$ related to the situation studied in this paper consists of three controls: the entrance control π_e, the remanufacturing control π_r, and the manufacturing control π_m.

The discounted expected cost over an infinite horizon for a policy $\boldsymbol{\pi}$, with $\mathbf{x} = (x_1, x_2)$ the state of the system when $t = 0$ and $\alpha > 0$ the discount rate, is given by :

$$v^{\boldsymbol{\pi}}(\mathbf{x}) = E\left[\int_0^{+\infty} e^{-\alpha t} C(\mathbf{X}(t))\mathrm{d}t \,|\{\mathbf{X}(0) = \mathbf{x}\ , \boldsymbol{\pi}\right]$$
$$+E\left[\sum_{i=1}^{\infty}\left(\begin{array}{c} e^{-\alpha\phi_a(i)}c^a + e^{-\alpha\phi_d(i)}c^b \\ +e^{-\alpha\phi_m(i)}c^m + e^{-\alpha\phi_r(i)}c^r \end{array}\right)|\mathbf{X}(0) = \mathbf{x}, \boldsymbol{\pi}\right].$$

where $\phi_a(i)$, $\phi_d(i)$, $\phi_m(i)$ and $\phi_r(i)$ respectively represent the i^{th} event time when either a return is accepted, a return is rejected, a product is manufactured or an accepted return is remanufactured. Knowing that the initial state is $(0, 0)$, we denote the average/discounted cost over an infinite horizon by

$$g(\boldsymbol{\pi}) = \begin{cases} v_\alpha^{\boldsymbol{\pi}}(0,0) & \text{if } \alpha > 0, \\ \lim_{a\to 0} a v_a^{\boldsymbol{\pi}}(0,0) & \text{if } \alpha = 0. \end{cases}$$

The objective is to minimize the expected average/discounted cost $g(\boldsymbol{\pi})$. This problem can be formulated as a continuous time Markov Decision Process (MDP). Let $v^\star$ be the optimal value function defined by $v^\star(x_1, x_2) = \min_{\boldsymbol{\pi}} \{v^{\boldsymbol{\pi}}(x_1, x_2)\} = v^{\boldsymbol{\pi}^\star}(x_1, x_2)$. The optimal policy is denoted by $\boldsymbol{\pi}^\star = (\pi_e^\star, \pi_r^\star, \pi_m^\star)$ and its related average/discounted cost by $g^\star = v^\star(0, 0)$. After uniformization of the MDP with rate $\tau = \lambda + \mu_r + \mu_m + \delta + \alpha$, we can transform the continuous time MDP into a discrete time MDP (Puterman, 1994). The optimal value function has to satisfy the following optimality equations:

$$v^\star = \mathcal{T} v^\star,$$

with

$$\mathcal{T} v(x_1, x_2) \quad = \quad \frac{1}{\tau} \begin{pmatrix} C(x_1, x_2) + \lambda v(x_1, x_2 - 1) + \delta T_e v(x_1, x_2) \\ +\mu_r T_r v(x_1, x_2) + \mu_m T_m v(x_1, x_2) \end{pmatrix}, \tag{4.1}$$

and

$$\begin{aligned} T_e v(x_1, x_2) &= \min\{v(x_1, x_2) + c^b, v(x_1 + 1, x_2) + c^a\}, \\ T_r v(x_1, x_2) &= \begin{cases} v(0, x_2) & \text{if } x_1 = 0, \\ \min\{v(x_1, x_2), v(x_1 - 1, x_2 + 1) + c^r\} & \text{otherwise.} \end{cases} \\ T_m v(x_1, x_2) &= \min\{v(x_1, x_2), v(x_1, x_2 + 1) + c^m\}. \end{aligned}$$

The operators T_e, T_r and T_m are related to respectively the entrance of returned remanufacturable products, the remanufacturing of remanufacturable products and the manufacturing of new products.

4.4 Characterization of the optimal policy

In order to prove that the optimal policy has some structural properties, we will show that the optimal value function is supermodular and superconvex (Koole, 1998).

Definition 4.4.1. $\forall v \in V$ and $\forall (x_1, x_2) \in \mathbb{Z} \times \mathbb{N}$:

(a) v is supermodular if and only if

$$v(x_1 + 1, x_2 + 1) - v(x_1 + 1, x_2) - v(x_1, x_2 + 1) + v(x_1, x_2) \geq 0,$$

(b) v is superconvex if and only if two conditions are respected:

$$\begin{cases} v(x_1 + 2, x_2) - v(x_1 + 1, x_2 + 1) - v(x_1 + 1, x_2) + v(x_1, x_2 + 1) \geq 0, \text{ and} \\ v(x_1, x_2 + 2) - v(x_1 + 1, x_2 + 1) - v(x_1, x_2 + 1) + v(x_1 + 1, x_2) \geq 0. \end{cases}$$

Note that supermodularity and superconvexity imply properties of convexity in directions x_1 and x_2:

$$\begin{cases} v(x_1 + 2, x_2) - 2v(x_1 + 1, x_2) + v(x_1, x_2) \geq 0, \text{ and} \\ v(x_1, x_2 + 2) - 2v(x_1, x_2 + 1) + v(x_1, x_2) \geq 0. \end{cases}$$

The following theorem is our main result about the existence and the structure of the optimal policy (the proof is given in Appendix B.1).

Theorem 4.4.1. *The optimal policy exists and it is a state-dependent base stock policy consisting of switching curves $S_m(x_1)$ for manufacturing, $S_r(x_1)$ for remanufacturing and $S_a(x_2)$ for acceptance of returns such that:*

- *produce at M_m if and only if $x_2 < S_m(x_1)$. Moreover $S_m(x_1) - 1 \leq S_m(x_1 + 1) \leq S_m(x_1)$.*
- *remanufacture at M_r if and only if $x_2 < S_r(x_1)$. Moreover $S_r(x_1) \leq S_r(x_1 + 1)$.*
- *accept returns in B_1 if and only if $x_1 < S_a(x_2)$. Moreover $S_a(x_2) - 1 \leq S_a(x_2 + 1) \leq S_a(x_2)$.*

So the optimal policy consists of three switching curves. Each switching curve divides the state space in two and is associated with one decision to take. To illustrate Theorem 4.4.1, we provide in Figure 4.3 the switching curves as a function of the state of the system for one set of parameter values.

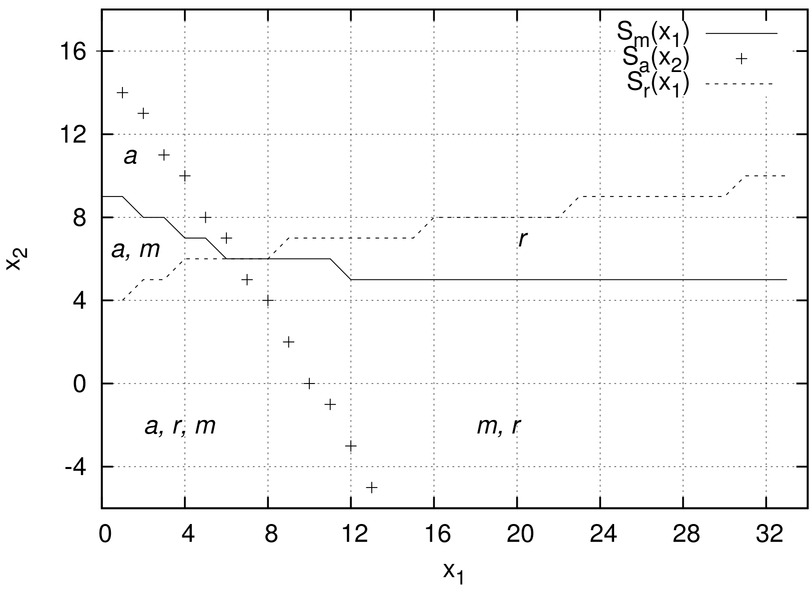


Figure 4.3: Average cost optimal policy for the instance $\{\delta = 0.6,\ \mu_r = 0.6,\ \mu_m = 0.6,\ \lambda = 1,\ h_1 = 1,\ h_2 = 5,\ b = 10\ ,\ c = 0\}$, with a, m, r respectively denoting the areas where the system accepts returns, manufactures, and remanufactures products.

When $h_1 \geq h_2$, we can derive the following additional result (see Appendix B.2).

Theorem 4.4.2. *If $h_1 \geq h_2$, the push remanufacturing policy is optimal.*

4.5 Direct reuse

Here we consider the special case, where accepted returns are immediately placed in the stock of serviceable products. This problem is a special case of the model with remanufacturing (see Section 4.3) because, for $\mu_r \to \infty$ (zero remanufacturing lead times) and $h_1 \geq h_2$ (remanufacture returned products as soon as possible is optimal), the remanufacturable product stock is zero and we can neglect it, resulting in Figure 4.4. To simplify the notations of this section, we denote x_2 by x and $X_2(t)$ by $X(t)$.

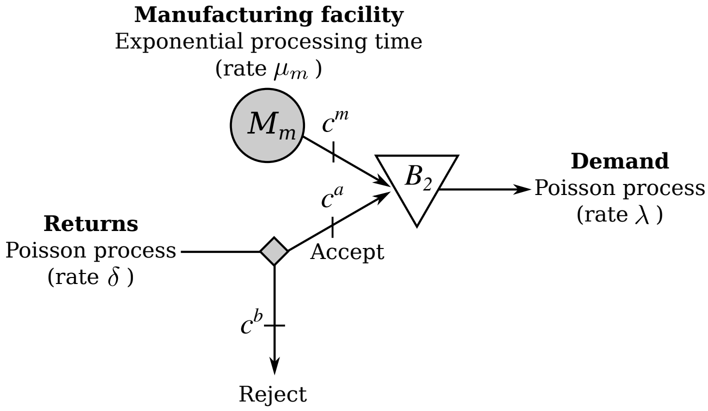


Figure 4.4: System without remanufacturing lead time.

In this case the MDP formulation becomes

$$T'v(x) = \frac{1}{\tau'} \left[C'(x) + \lambda v(x-1) + \mu_m T'_m v(x) + \delta T'_e v(x+1) \right] \tag{4.2}$$

with the uniformization rate $\tau' = \alpha + \lambda + \mu_m + \delta$, the cost function $C'(x) = hx^+ + bx^-$, the entrance operator $T'_e = \min\{v(x) + c^b, v(x+1) + c^a\}$, and the manufacturing operator $T'_m = \min\{v(x), v(x+1) + c^m\}$. Because T'_e, T'_m, and C propagate convexity (Koole, 1998), the unique solution of the equation $v'^{\star} = T'v'^{\star}$ is convex. So the structure of the optimal policy is a two-threshold (S_m, S_a) policy.

Then, $\{X(t)\}$ can be described by a continuous Markov chain with a birth death process structure.

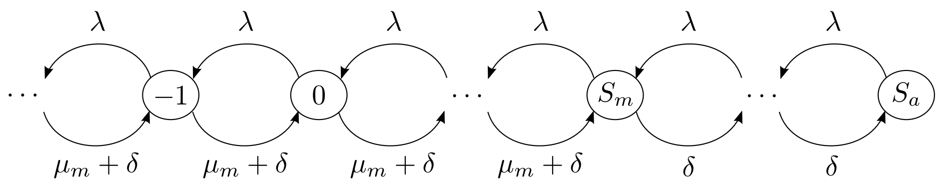


Figure 4.5: Markov chain of the single echelon problem with $c^a < c^m + c^b$.

This result is consistent with the results of the model described in Section 4.3. The optimal policy for the later is a three-threshold policy $(S_m(x_2), S_a(x_1), S_r(x_1))$. Here, with no remanufacturing, we can set $x_1 = 0$ and $S_r(0) = \infty$ because the remanufacturing server produces returned products as soon as possible, and does this with an infinite service rate. So the queue B_1 is always empty and the optimal policy reduces to a (S_m, S_a) policy.

Because the related Markov chain is a simple birth death process, we can write down the stationary probabilities and derive an analytical expression for the average cost with $\rho = \lambda/\mu_m$ and $\rho_r = \lambda/(\mu_m + \delta)$.

This problem is already considered by Zerhouni et al. (2009). Albeit correct, their results were not fully justified, because they made the implied assumption that one of the parameters of the optimal policy is positive. We present their main results in the following, the full proof and the justification of their implied assumption are given in appendix B.5.

Case 1 : $c^a < c^m + c^b$

The optimal manufacturing base stock level $S_m^\star(w)$ for a given $w = S_m - S_a$ $(w \geq 0)$ is given by,

$$S_m^*(w) = \min\left\{s|F(s) \geq \frac{b}{b+h}\right\}$$

with

$$F(S_m) = \frac{(1-\rho_r)(1-\rho^w) + \rho^w(1-\rho_r^{S_m - w+1})(1-\rho)}{1-\rho_r - \rho^w(\rho - \rho_r)}.$$

Case 2: $c^a > c^m + c^b$

The optimal entrance base stock level $S_a^\star(w)$ for a given $w = S_a - S_m$ $(w \geq 0)$ is given by,

$$S_a^*(w) = \min\left\{s|F(s) \geq \frac{b}{b+h}\right\}$$

with

$$F(S_a) = \frac{(1-\rho_r)(1-\rho^w) + \rho^{w-1}(1-\rho_r^{S_a - w+1})(\rho-1)}{1-\rho_r - \rho^{-w}(1/\rho - \rho_r)}.$$

Case 3: $c^a = c^m + c^b$

$$S_a^* = S_m^* = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \rho_r} \right\rfloor$$

4.6 Heuristic policies

The deviation between a heuristic policy and the optimal policy is given by

$$\Delta g(\boldsymbol{\pi}) = \frac{g(\boldsymbol{\pi}) - g^\star}{g^\star}.$$

Note that, for the policy $(\pi_e, \pi_r^\star, \pi_m^\star)$, called heuristic entrance control policy, the controls of manufacturing and remanufacturing are optimally made given that the entrance control is heuristic. Moreover, all the parameters of the heuristic controls are optimized to minimize the cost over an infinite horizon (see the computational procedure in Appendix B.4).

In the following we use an extensive numerical study to evaluate the performances of the policies. We compute the cost of several heuristic policies, for each of the 6160 combinations of

$$\alpha \in \{0.1, 0\}, \lambda = 1, \delta \in \{0.2, 0.5, 0.8, 1.1\},$$
$$\mu_r \in \{0.2, 0.5, 1, 2\}, \mu_m \in \{0.2, 0.5, 1, 2\},$$
$$c^b = c^r = 0, c^m \in \{0, 5, 10\}, c^a \in \{0, 5, 10\},$$
$$h_1 = 1, h_2 \in \{1.5, 5, 10\}, b \in \{2, 10, 100\},$$

which satisfy the stability condition $\lambda < \mu_m + \min\{\mu_r, \delta\}$. According to Appendix B.3, we can set $c^b = c^r = 0$, $\lambda = 1$ and $h_1 = 1$ without loss of generality. Then, for all the instances and for a given heuristic policy $\boldsymbol{\pi}$, we denote the average deviation by $\overline{\Delta g(\boldsymbol{\pi})}$, the maximal deviation by $\max\{\Delta g(\boldsymbol{\pi})\}$, and the percentage of the instances with deviation lower than 1% by $\Delta g(\boldsymbol{\pi}) < 1\%$.

We consider several types of heuristic policies. In Section 4.6.1 we focus on heuristic entrance controls, other decisions being made optimally $(\pi_e, \pi_r^\star, \pi_m^\star)$. In the same way, we consider in Sections 4.6.2 (resp. 4.6.3) policies with heuristic control of remanufacturing (resp. manufacturing) and optimal control of the other decisions. In this way, we show the influence of each heuristic control separately. More precisely, we show numerically that controlling optimally manufacturing and return entrance may bring a significant saving, that the rejecting option greatly increases the performance of the system, and that the choice of a powerful heuristic control mainly depends of the server capacity constraints. Finally, in Section 4.6.4 we consider four heuristic policies, with simultaneously heuristic control of entrance, remanufacturing, and manufacturing, three of them adapted from literature and one new. We compare them with the optimal policy and show the performance and the robustness of the new heuristic policy.

4.6.1 Entrance control of returns

The switching curve for the acceptance of returns is described in Figure 4.6. The two extreme cases for this switching curve are the decreasing diagonal switching curve (accept returns if $x_1 + x_2 < S_e$) and the vertical switching curve (accept returns if $x_1 < S_e$) represented with doted lines in Figure 4.6.

In the following we limit us to the heuristic entrance controls consistent with the extreme cases of the optimal policy:

$[\boldsymbol{x_1 + x_2}]$: Accept returns if $x_1 + x_2 < S_e$. This control is used by Simpson (1978). Note that $x_1 + x_2$ takes into account the current number of products in the system minus the current backlogged demands.

$[\boldsymbol{x_1 + x_2^+}]$: Accept returns if $x_1 + x_2^+ < S_e$. This control is used by Gupta and Korugan (2000). Note that $x_1 + x_2^+$ takes into account the current number of products in the

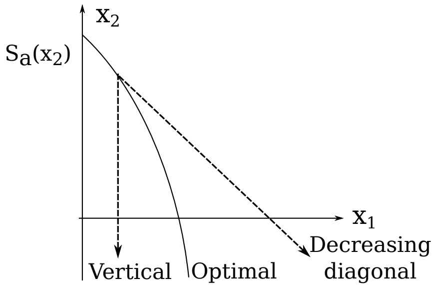


Figure 4.6: Slope of the optimal acceptance switching curve.

system without taking into account backlogs.

$[\boldsymbol{x_1}]$: Accept returns if $x_1 < S_e$. This control is used by van der Laan et al. (1996b) and van der Laan and Salomon (1997). Note that x_1 takes into account the current number of products in the remanufacturable stock.

$[\boldsymbol{rej}]$: Reject all returns reduces our problem to an M/M/1 make-to-stock queue studied by Dusonchet and Hongler (2003) and Veatch and Wein (1996).

$[\boldsymbol{acc}]$: Accept all returns is a control used by van der Laan and Teunter (2006) and Kiesmüller (2003).

The acceptance controls $[acc]$ and $[rej]$ are special cases of the entrance controls $[x_1 + x_2]$, $[x_1 + x_2^+]$, and $[x_1]$. For instance to accept if $x_1 < S_e$ is equivalent to accept all returns with $S_e \to \infty$ and to reject all returns with $S_e \to -\infty$. So,

$$g^\star \quad \leq \quad \begin{cases} g\left(\pi_a^{[x_1+x_2]}, \pi_r^\star, \pi_m^\star\right), \\ g\left(\pi_a^{[x_1+x_2^+]}, \pi_r^\star, \pi_m^\star\right), \\ g\left(\pi_a^{[x_1]}, \pi_r^\star, \pi_m^\star\right) \end{cases} \quad \leq \quad \begin{cases} g\left(\pi_a^{[acc]}, \pi_r^\star, \pi_m^\star\right), \\ g\left(\pi_a^{[rej]}, \pi_r^\star, \pi_m^\star\right). \end{cases}$$

The table 4.1 gives the result of the extensive numerical study comparing the entrance control policies. We observe that optimally controlling the entrance of returns can make a significant difference. Among the 6160 instances of the extensive study, at least one instance meets $\Delta g > 7\%$ for every heuristic entrance control policy. Note that the infinite values for $[rej]$ and $[acc]$ are due to the system instability induced by these policies. Controlling the entrance of returns with $[x_1]$ is interesting because it considers only local data (we control entrance in stock B_1 only with the level of stock in B_1) but we observe numerically that it performs poorly.

The main criterion to choose between $[x_1 + x_2]$, and $[x_1 + x_2^+]$ is the ratio of μ_r and δ. Without remanufacturing and manufacturing capacity constraints and with equal lead times, Simpson (1978) proves that controlling return entrance with $[x_1 + x_2]$ is optimal. In our case, no capacity corresponds to infinite μ_m and μ_r. In Figure 4.7 we observe

π_e	$\overline{\Delta g\,(\pi_e, \pi_r^\star, \pi_m^\star)}$	$\max\{\Delta g\,(\pi_e, \pi_r^\star, \pi_m^\star)\}$	$\Delta g\,(\pi_e, \pi_r^\star, \pi_m^\star) < 1\%$
$[x_1 + x_2]$	1.00%	26.4%	76.7%
$[x_1 + x_2^+]$	1.02%	17.6%	71.9%
$[x_1]$	1.75%	25.3%	53.6%
$[rej]$	∞	∞	8.66%
$[acc]$	∞	∞	13.5%

Table 4.1: Comparison of heuristic entrance control policies with the optimal policy via the extensive numerical study.

that $\mu_r \gg \delta$ on remanufacturing suffices to make the entrance control with $[x_1 + x_2]$ near optimal. Inversely, when the remanufacturing capacity constraint is strong (i.e. $\mu_r \ll \delta$) the control $[x_1 + x_2^+]$ improves. Indeed, when there are several backlogged demands, to add another return in the stock B_1 is not necessary, since the remanufacturing facility will not process it quickly. Therefore, when the remanufacturing facility is the bottleneck, $[x_1 + x_2^+]$ will be preferred.

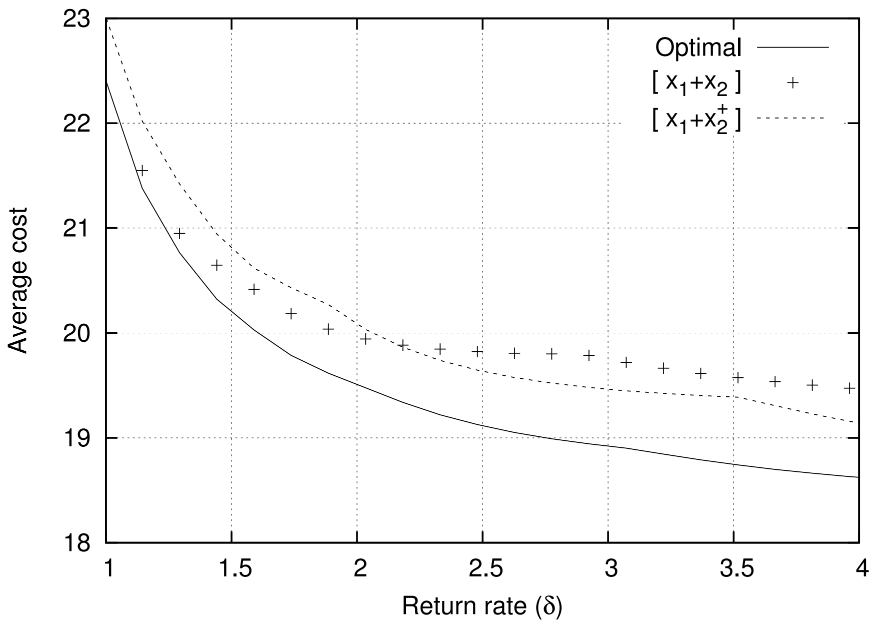


Figure 4.7: Average cost $g(\pi_e, \pi_r^\star, \pi_m^\star)$ in function of δ for the instance $\{\mu_r = 1,\ \mu_m = 0.5,\ \lambda = 1,\ h_1 = 1,\ h_2 = 10,\ b = 10,\ c^a = c^r = 0 = c^m = c^b = 0,\ \alpha = 0\}$.

The performances of the $[acc]$ and $[rej]$ returns acceptance controls are generally poor. The main criterion to select $[acc]$ vs. $[rej]$ is the sign of $c^a + c^r - c^b - c^m$. If $c^a + c^r - c^b > c^m$, like in the first line of the Table 4.2, (resp. $c^a + c^r - c^b < c^m$, third line of the Table 4.2), the net remanufacturing cost $\tilde{c}^r = c^a + c^r - c^b$ is higher (resp. lower) than the manufacturing cost, so the $[rej]$ control should (resp. should not) be preferred.

c^a	c^m	$g^\star$	$g\left(\pi_a^{[acc]}, \pi_r^\star, \pi_a^\star\right)$	$g\left(\pi_a^{[rej]}, \pi_r^\star, \pi_a^\star\right)$
50	0	10.3	23.8	13.5
25	25	30.5	36.7	38.5
0	50	22.4	23.3	63.5

Table 4.2: Average cost of systematic entrance controls in function of c^a and c^m. With $\delta = 0.8$, $\mu_r = 0.9$, $\mu_m = 1.2$, $\lambda = 1$, $h_2 = 2$, $b = 4$, $c^r = 1$, $c^b = 2$, and $\alpha = 0$.

4.6.2 Control of remanufacturing

The remanufacturing switching curve is increasing (see Figure 4.8). The two extreme cases are the horizontal switching curve (remanufacture if $x_2 < S_r$) and the vertical switching curve (remanufacture if $x_1 > S_r$).

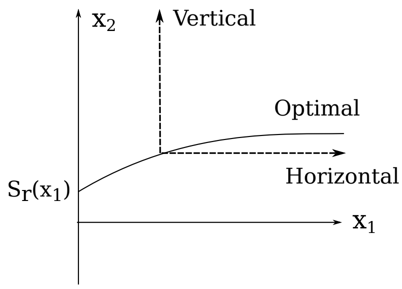


Figure 4.8: Slope of the optimal remanufacturing switching curves.

In the following we limit ourselves to the heuristic remanufacturing controls consistent with the extreme cases of the optimal policy:

$[\boldsymbol{x_2}]$: Remanufacture if $x_2 < S_r$. This heuristic control is used by Simpson (1978); van der Laan and Salomon (1997); Gupta and Korugan (2000); van der Laan and Teunter (2006) and Kiesmüller (2003). Note that the variable x_2 is the current number of products in the serviceable stock minus the current backlogged demand.

$[\boldsymbol{x_1}]$: As far as we know, remanufacture when $x_1 > S_r$ is never used in literature.

[***push***] : Remanufacture always (if $x_1 > 0$) is a control used by van der Laan et al. (1996a,b); van der Laan and Salomon (1997) and van der Laan and Teunter (2006).

Note that we do not consider the [*never*] remanufacture control because it reduces to an M/M/1 make-to-stock queue, so it is equivalent to control entrance with [*rej*] (see section 4.6.1).

Because the [*push*] remanufacturing control is a special case of $[x_1]$ remanufacturing controls (with $S_r = -1$) and $[x_2]$ (with $S_r \to \infty$), we can write:

$$g^\star \quad \leq \quad \begin{cases} g\left(\pi_a^\star, \pi_r^{[x_1]}, \pi_m^\star\right), \\ g\left(\pi_a^\star, \pi_r^{[x_2]}, \pi_m^\star\right) \end{cases} \quad \leq \quad g\left(\pi_a^\star, \pi_r^{[push]}, \pi_m^\star\right).$$

Note that we proved in Section 4.4 that $[push]$ remanufacturing control is optimal if $h_1 \geq h_2$, so in this case, $[x_1]$ and $[x_2]$ are optimal too.

Table 4.3 gives the result of the extensive numerical study comparing the heuristic remanufacturing control policies with the optimal policy. We observe that $[x_2]$ performs always well, and the performance of $[x_1]$ and $[push]$ seems equivalent.

π_r	$\overline{\Delta g(\pi_e^\star, \pi_r, \pi_m^\star)}$	$\max\{\Delta g(\pi_e^\star, \pi_r, \pi_m^\star)\}$	$\Delta g(\pi_e^\star, \pi_r, \pi_m^\star) < 1\%$
$[x_2]$	0.06%	3.41%	99.0%
$[x_1]$	4.70%	135%	59.4%
$[push]$	4.70%	135%	59.4%

Table 4.3: Comparison of heuristic remanufacturing control policy with the optimal policy via the extensive study.

This observation is confirmed in Figure 4.9 which shows that controlling the remanufacturing with data $[x_2]$ is near optimal.

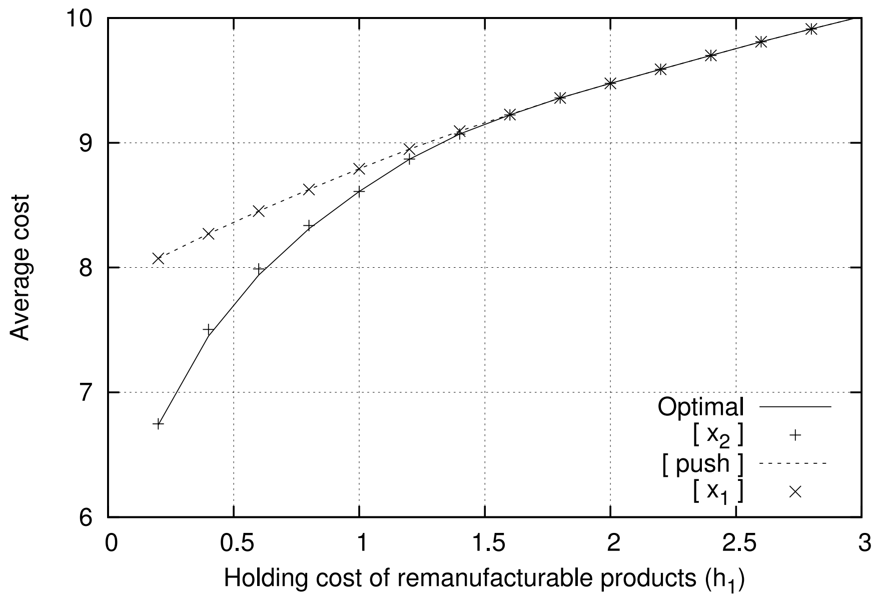


Figure 4.9: Average cost $g(\pi_e^\star, \pi_r, \pi_m^\star)$ in function of h_1 for the instance $\{\delta = 0.7,\ \mu_r = 1.2,\ \mu_m = 0.8,\ \lambda = 1,\ h_2 = 2,\ b = 4,\ c^a = c^r = 1,\ c^m = c^b = 2,\ \alpha = 0\}$.

4.6.3 Control of manufacturing

The structure of the optimal manufacturing switching curve is decreasing as described in Figure 4.10. The two extreme cases are horizontal switching curve (manufacture if $x_2 < S_m$) and decreasing diagonal switching curve (manufacture if $x_1 + x_2 < S_m$).

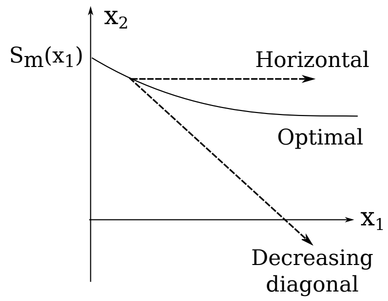


Figure 4.10: Slope of the manufacturing switching curves.

In the following we limit ourselves to the heuristic manufacturing controls consistent with the extreme cases of the optimal policy:

[$\boldsymbol{x_2}$] : Manufacture if $x_2 < S_a$. This control is used by van der Laan and Salomon (1997); Gupta and Korugan (2000), and van der Laan and Teunter (2006).

[$\boldsymbol{x_1 + x_2}$] : Manufacture if $x_1 + x_2 < S_a$. This control is used by Simpson (1978); van der Laan et al. (1996a,b), and Kiesmüller (2003).

[*alw*] : As far as we know, always manufacture is never used in situations with remanufacturing.

[*nev*] : Never use the manufacture facility reduces the problem to the make-to-stock tandem queue model studied by Veatch and Wein (1992, 1994).

Because $[alw]$ (resp. $[nev]$) is a special case of $[x_2]$ and $[x_1 + x_2]$ based manufacturing control policies with $S_a \to \infty$ (resp. $S_a \to -\infty$), we know that,

$$g^\star \quad \leq \quad \begin{cases} g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[x_2]}\right), \\ g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[x_1+x_2]}\right) \end{cases} \quad \leq \quad \begin{cases} g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[alw]}\right), \\ g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[nev]}\right). \end{cases}$$

Table 4.4 gives the result of the extensive numerical study comparing the different manufacturing control policies with the optimal policy. We observe that optimally controlling the manufacturing can make a significant difference. Moreover, in the 6160 instances of the extensive study, at least one instance meets $\Delta g > 6\%$ for every heuristic manufacturing control policies considered.

Once again, the main choice criterion for choosing which manufacturing control is the ratio between μ_r and δ. Simpson (1978) proves that the optimal parameter to control

π_m	$\overline{\Delta g\,(\pi_e^\star, \pi_r^\star, \pi_m)}$	$\max\{\Delta g\,(\pi_e^\star, \pi_r^\star, \pi_m)\}$	$\Delta g\,(\pi_e^\star, \pi_r^\star, \pi_m) < 1\%$
$[x_2]$	0.39%	9.82%	87.9%
$[x_1 + x_2]$	1.03%	23.7%	69.9%
$[alw]$	∞	∞	5.94%
$[nev]$	∞	∞	0.00%

Table 4.4: Comparison of heuristic entrance controls with the optimal policy in an extensive study.

manufacturing and entrance of returns is $[x_1 + x_2]$ if there is no capacity constraint and equal lead times for manufacturing and remanufacturing. A numerical control (see Figure 4.11) shows that a small capacity constraint on remanufacturing ($\mu_r \gg \delta$) makes the control $[x_1 + x_2]$ powerful. If $\mu_r \to \infty$ (i.e. zero remanufacturing lead times), a remanufacturable product can be available immediately to serve demand. In this case, stocks B_1 and B_2 can be merged and then, we control the entrance of new product returns with the data $[x_1 + x_2]$. Inversely, with a long remanufacturing time ($\mu_r \ll \delta$), a remanufacturable product is not available to serve demand before a long time, so we control manufacturing with data $[x_2]$.

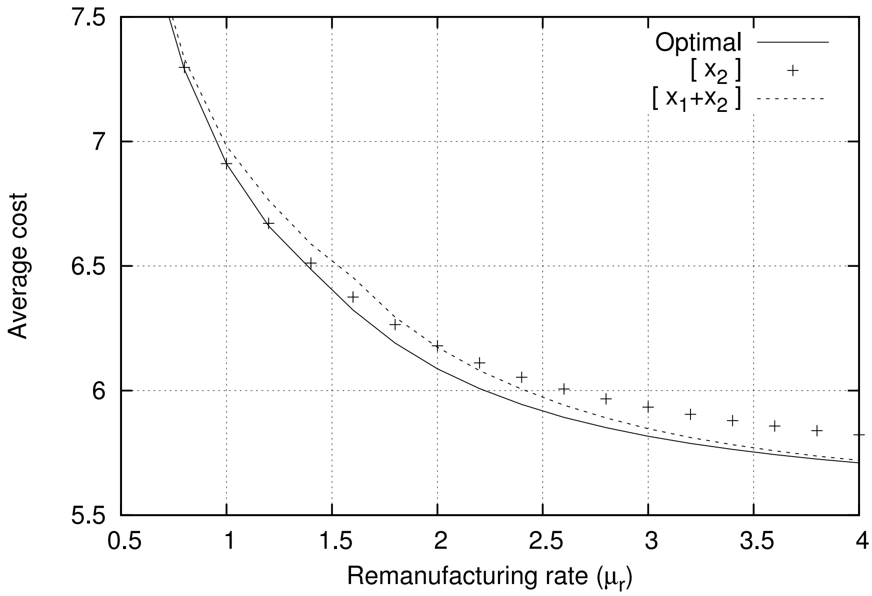


Figure 4.11: Average cost $g(\pi_e^\star, \pi_r^\star, \pi_m)$ in function of μ_r for the instance $\{\delta = 0.7,\ \mu_m = 1.2,\ \lambda = 1,\ h_1 = 1,\ h_2 = 2,\ b = 4,\ c^a = c^r = 1,\ c^m = c^b = 2,\ \alpha = 0\}$.

Without data, possible controls are $[alw]$ and $[nev]$. The main criterion to select $[alw]$ instead of $[nev]$ is the same as for $[acc]$ vs. $[rej]$. The sign of the net remanufacturing cost $\tilde{c}^r$ defines the preference of remanufacturing flow over the manufacturing flow or the contrary. Table 4.5 gives some instances of this behavior. We can observe that the performance of both controls is generally weak.

c^a	c^m	$g^\star$	$g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[alw]}\right)$	$g\left(\pi_e^\star, \pi_r^\star, \pi_m^{[nev]}\right)$
0	50	14.8	44.2	23.5
25	25	32.1	34.3	48.5
50	0	24.3	24.3	73.5

Table 4.5: Average cost of systematic manufacturing controls in function of c^a and c^m. With $\delta = 1.2$, $\mu_r = 1.2$, $\mu_m = 0.6$, $\lambda = 1$, $h_2 = 2$, $b = 4$, $c^r = 1$, $c^b = 2$, and $\alpha = 0$.

4.6.4 Joint heuristic strategies

In this section we consider heuristic policies from literature simultaneously using heuristic control for entrance, remanufacturing, and manufacturing.

First, we observe that the gaps resulting from using three heuristic control policies are almost additive (see Figure 4.12),

$$\Delta g(\pi_e, \pi_r, \pi_m) \approx \begin{pmatrix} \Delta g(\pi_e, \pi_r^\star, \pi_m^\star) \\ +\Delta g(\pi_e^\star, \pi_r, \pi_m^\star) \\ +\Delta g(\pi_e^\star, \pi_r^\star, \pi_m) \end{pmatrix} = \Delta g^+(\pi_e, \pi_r, \pi_m).$$

It justifies the previous approach which considered the effect of using a heuristic control for one type of decision only.

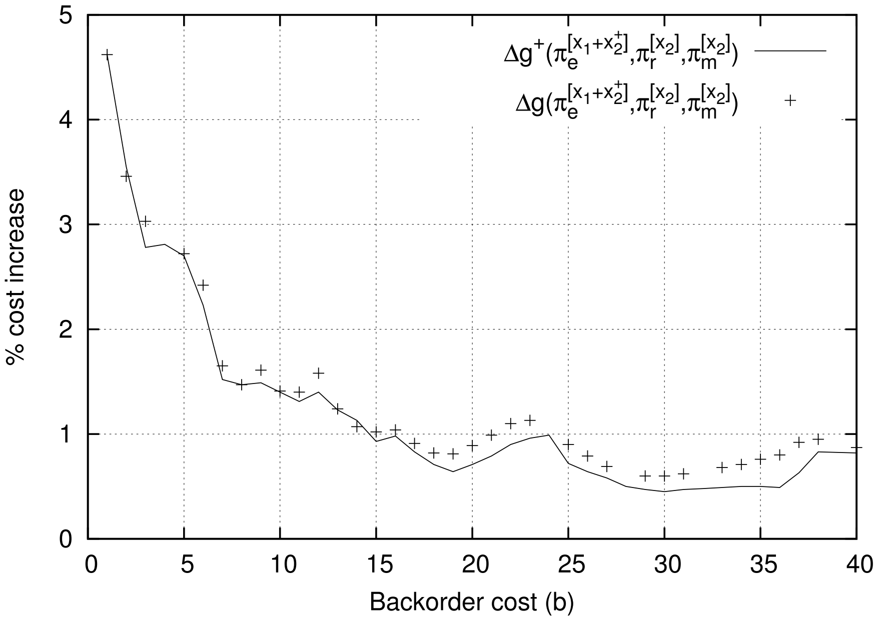


Figure 4.12: Percentage cost increase for the instance $\{\delta = 0.8$, $\mu_r = 0.8$, $\mu_m = 0.8$, $\lambda = 1$, $h_1 = 1$, $h_2 = 5$, $c^a = 1$, $c^r = 3$ $c^m = 1$ $c^b = 3\}$.

Now, we adapt the heuristic policies found in literature (see Section 4.2) to the situation studied in this paper. Since the set-up cost is zero, we set the lot size Q to 1, and since preemption is allowed, we replace I (the inventory position of serviceable products) by x_1 and R (the inventory position of remanufacturable products) by x_2. Considering that systematic heuristic controls ([*push*], [*acc*],...) performs poorly, we are only interested by the heuristic policies with non-systematic heuristic controls ($[x_1 + x_2^+]$, $[x_1]$,...) on entrance, remanufacturing and manufacturing. We call the three heuristic policies from the literature meeting this condition: Base Stock Return (BSR), Fixed Buffer (FB), and Kanban (KB) policies (see Table 4.6).

We add a new heuristic policy based on Theorem 4.4.1, the Base Stock Echelon (BSE) policy. It controls the entrance with the data $x_1 + x_2$ and controls manufacturing and remanufacturing with the data x_2. This policy is known to be optimal for a tandem queue without capacity and without set-up costs and it generates good results for the capacitated problem (Veatch and Wein, 1994; Parker and Kapuscinski, 2004).

Note that the heuristic policies described in Table 4.6 are consistent with the results about preferred heuristic controls fund in the previous sections: in each of them, the entrance controls are $[x_1 + x_2]$, $[x_1 + x_2^+]$ or $[x_1]$, the remanufacturing control is $[x_2]$, and the manufacturing controls are $[x_1 + x_2]$ or $[x_2]$.

Table 4.7 gives the result of the extensive numerical study for these heuristic policies. We observe that, for the instances considered, the base stock echelon policy has the best performance with an average deviation of 1.58%, and a maximal deviation of 26.5% when compared to the optimal policy. Note that the Kanban policy obtains good results too. We recommend to select the Kanban policy if the server of remanufacturing is clearly the bottleneck, and the base stock echelon policy otherwise.

4.7 Extension : disposal option

In this section we study the impact of the possibility to dispose returns after acceptation at any time with a unit cost c^d. We consider two cases: a problem with direct reuse of returns and a hybrid problem with manufacturing and remanufacturing.

4.7.1 Direct reuse

We assume for the single stage problem that serviceable products can be disposed (see Figure 4.13).

We characterize the structure of the optimal policy in the following theorem.

Theorem 4.7.1. *The optimal policy is a three threshold (S_m, S_a, s_d) policy, which produces only when $x < S_m$, accepts return only when $x < S_a$, and dispose, at any time, a quantity $(x - s_d)^+$ with $s_d \geq 0$.*

Moreover, for all time $t > 0$,

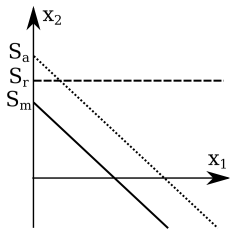
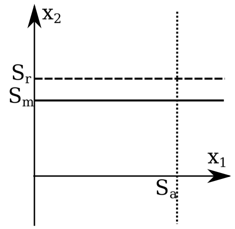
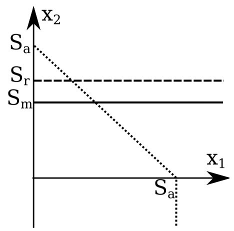
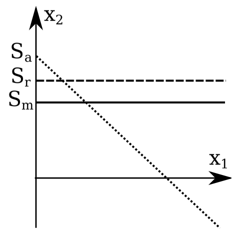
Policies	Base stock ret. (BSR) (x_1+x_2, x_2, x_1+x_2)	Fixed buffer (FB) (x_1, x_2, x_2)	Kanban (KB) $(x_1, x_1+x_2^+, x_2)$	Base stock ech. (BSE) (x_1, x_1+x_2, x_2)
Adapted from	Simpson (1978)	van der Lann and Salomon (1997)	Gupta and Korugan (2000)	NEW
Accept. if	$x_1+x_2 < S_a$	$x_1 < S_a$	$x_1+x_2^+ < S_a$	$x_1+x_2 < S_a$
Reman. if	$x_2 < S_r$	$x_2 < S_r$	$x_2 < S_r$	$x_2 < S_r$
Manuf. if	$x_1+x_2 < S_m$	$x_2 < S_m$	$x_2 < S_m$	$x_2 < S_m$
Switching curves				

Table 4.6: Heuristic policies

Policies	$\overline{\Delta g}$	$\max\{\Delta g\}$	$\Delta g < 1\%$
BSR	4.02%	67.4%	37.3%
FB	2.36%	32.4%	36.2%
KB	1.67%	29.3%	52.9%
BSE	1.58%	26.5%	58.6%

Table 4.7: Comparison of heuristic policies with the optimal policy via the extensive numerical study.

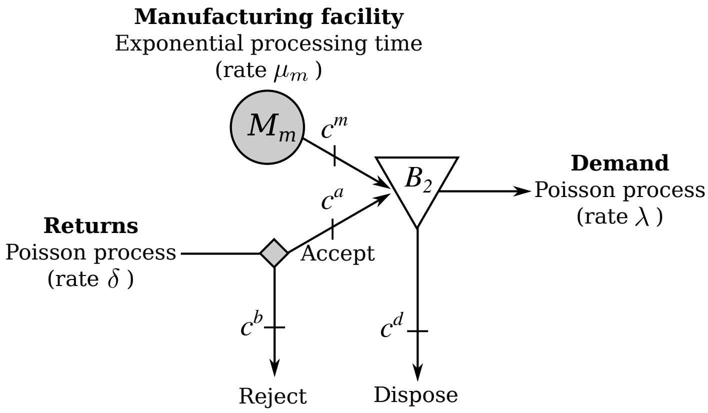


Figure 4.13: System with direct reuse

- *if $c^d + c^a < c^b$, all returns are accepted and can be dispose upon arrival. Furthermore, items are disposed one by one only when a return event occurs.*
- *if $c^d + c^a > c^b$, the disposal option is never used, accepted return are send to clients.*
- *if $c^d + c^a = c^b$, dispose and reject are equivalent.*

The proof of this theorem is given is section B.6.1.

Corollary 1. *The single stage problem with disposal and rejection options can always be reduced to an equivalent (same optimal policy) system without a disposal option.*

Proof. If $c^d + c^a \geq c^b$ the disposal option is never used, so the system can be reduced. If $c^d + c^a < c^b$ the rejection option is never used, so the system has only a disposal option. But the disposal option occurs only with returns events, so it is equivalent to a return entrance control. Then, the systems $\{\lambda, \mu_m, \delta, h, b, c^m, c^r, c^a, c^b, c^d, x_0\}$ and $\{\lambda, \mu_m, \delta, h, b, c^m, c^r, c^a, \widetilde{c}_r = \min\{c^b - c^a, c^d\} + c^a, \widetilde{c}_d = 0, \widetilde{x}_0 = \min\{x_0, s_d\}\}$ are equivalent. $\square$

4.7.2 Remanufacturing

We stress the fact that a single stage system with disposal is not a particular case of our two-stage problem since the serviceable products can not be disposed.

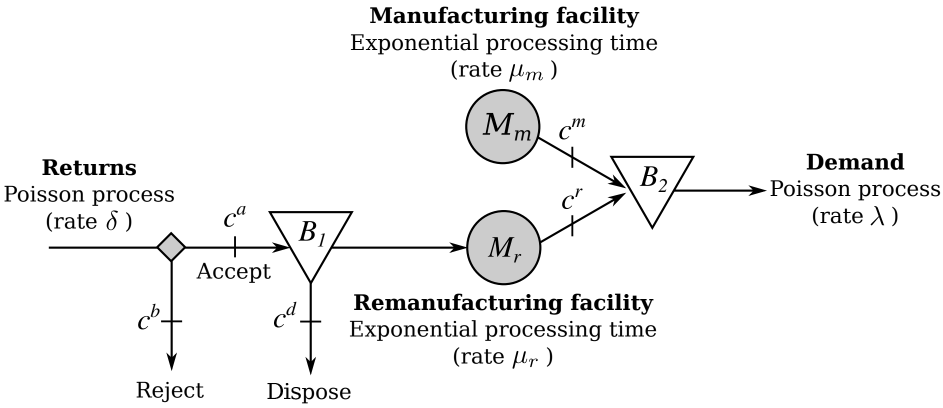


Figure 4.14: Two stage system

Theorem 4.7.2. *The optimal policy is a four-threshold* $(S_m(x_1), S_r(x_1), S_a(x_2), s_d(x_2))$ *policy, which produces only when* $x_2 < S_m(x_1)$ *, remanufacture only when* $x_2 < S_r(x_1)$*, accepts return only when* $x_1 < S_a(x_2)$*, and dispose* $(x_1 - s_d(x_2))^+$ *items.*

Moreover, for all positive time $(t > 0)$*, products are disposed one by one, and*

- *if* $c^d + c^a \geq c^b$*, the disposal option can be taken only on manufacturing event.*

- *if* $c^d + c^a \leq c^b$*, all returns are accepted and the disposal option is taken only on manufacturing event and return event.*

The proof of this theorem is given is section B.6.2. Because there is no analytical reduction of the two stages disposal problem, we propose a numerical study to quantify the gain of the disposal option. With the following parameters values,

$$\alpha \in \{0.1, 0\}, \lambda = 1, \delta \in \{0.2, 0.5, 0.8, 1.1\},$$
$$\mu_r \in \{0.2, 0.5, 1, 2\}, \mu_m \in \{0.2, 0.5, 1, 2\},$$
$$c^m = c^r = 0, c^a = 5, c^b \in \{0, 5, 10\}, c^d \in \{0, 5, 10\},$$
$$h_1 = 1, h_2 \in \{1, 2, 10\}, b \in \{2, 10, 100\},$$

we compute all the 6177 stable instances (with $\lambda < \min\{\delta, \mu_r\}$). We compare the optimal policy (OP) with two heuristic policies, when the system can not dispose (*noDispose*) and when the system can not reject (*noReject*).

Table 4.8 gives some results of this study, with $\Delta g =$ (heuristic $- OP)/OP$.

Obviously, in the cases $c^b > c^d + c^a$ (resp. $c^b < c^d + c^a$) the disposal (resp. rejection) option is very important, because the rejection option is never used. In the cases $c^b = c^d + c^a$, the disposal option can save maximum 2.51% of the cost, but the average gain is 0.04%, so we can neglect the disposal option.

Instances		noDispose (%)	noReject (%)
	$\Delta g < 1\%$	29.0	100
$c^b > c^d + c^a$ (2059 instances)	$\overline{\Delta g}$	13.9	0
	$\max\{\Delta g\}$	150	0
	$\Delta g < 1\%$	100	29.3
$c^b < c^d + c^a$ (2059 instances)	$\overline{\Delta g}$	0.00	13.7
	$\max\{\Delta g\}$	0.02	146
	$\Delta g < 1\%$	99.6	100
$c^b = c^d + c^a$ (2059 instances)	$\overline{\Delta g}$	0.04	0
	$\max\{\Delta g\}$	2.51	0

Table 4.8: Extensive numerical study to evaluate the benefits linked to the disposal option and the rejection option.

4.8 Conclusion

In this paper we studied a production-inventory system with return acceptance, and capacitated manufacturing and remanufacturing with stochastic non-zero process times. We proved that the structure of the optimal policy is a state dependent base stock policy with three thresholds. Numerical study allowed us to show that the return entrance control substantially increases the performance. Moreover, we showed that the optimal control of manufacturing and entrance brings a significant saving. We compared the optimal policy with four heuristic policies (base stock return, fixed buffer, Kanban, and base stock echelon). The base stock echelon policy was numerically demonstrated to be the best performance policy among others. However, when selecting a heuristic policy, we recommend to use Kanban policy if the server of remanufacturing is clearly the bottleneck and the base stock echelon policy otherwise. An extension presents the possibility of dispose returns after acceptance. In the case of direct reuse and remanufacturing the optimal policy is characterized. We observed numerically that the dispose option can be neglected if the cost of dispose and the relative cost of rejection are equals.

Several avenues can be considered for further research. Among them: an additional stage of manufacturing or a shared facility of manufacturing and remanufacturing are among them. Assumptions like no set-up cost, preemption allowed and exponentially distributed lead times could be relaxed.

Chapter 5

Control a production-inventory system with an imperfect advance return information

This work was initiated during the PhD of Hichem Zerhouni, who conducted a part of the numerical analysis and derived the results of Section 5.3.1.

We consider a production-inventory system with product returns that are announced in advance by the customers. Demands and announcements of returns occur according to independent Poisson processes. An announced return is either actually returned or cancelled after a random return lead time. We consider both lost sale and backorder situations. Using a Markov decision formulation, the optimal production policy, with respect to the discounted cost over an infinite horizon, is characterized for situations with and without advance return information. We give insights in the potential value of this information. Also some attention is paid to combining advance return and advance demand information. Further applications of the model as well as topics for further research are indicated.

5.1 Introduction

During the last 15 years a lot of attention has been paid to so called closed-loop supply chains, reverse logistics, product recovery, both in practice, as in academic literature, see e.g. Dekker (2004) and Rubio et al. (2008). In this context, also attention has been paid to forecasting the reverse flows. Available publications use delivery/purchase information to forecast returns, see e.g. Yuan and Cheung (1998), sometimes taking into account information on actual returns, see e.g. de Brito and van der Laan (2009).

In this paper we neglect the use of the above information, but focus on return information supplied by the owner/user of a product after the initial delivery, purchase of this product. We study situations where customers have to announce the return of a product (see 5.1). Advance Return Information/Advance Supply Information (ARI/ASI) is among others required in practice for warranty returns, commercial / buy back contract returns, returns due to wrong delivery. An important reason for the above is to prevent unnecessary or incorrect returns. See e.g. Boykin (2001) for a general description of the Return Material Authorization process and the support offered for this process by SAP. Other examples of using ARI concern information related to the end of lease contracts, when the lessee has to indicate some time before whether or not (s)he will continue the contract or buy the leased product.

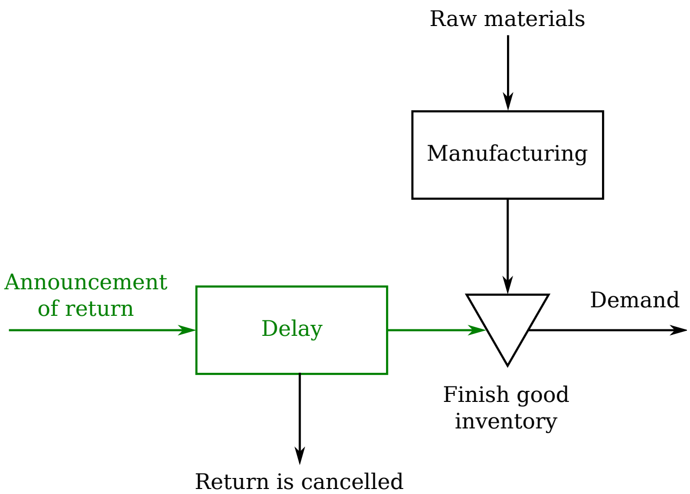


Figure 5.1: Single stage production/inventory system with announced returns.

A number of authors paid already attention to the value of advance information in the context of product recovery, including the recent contribution by Khawam and Hausman (2009) with an up-to-date review of the literature in this field. Our paper differs from the above paper in a number of aspects including the origin of supply uncertainty, a finite production capacity, a continuous review of the inventory position, random leadtimes and lost sales.

We adopt a make-to-stock queue framework to model production capacity and uncertainty with respect to production, returns and demand. A make-to-stock queue refers to a make-to-stock system where the supply process is modeled by servers producing units one by one. Make-to-stock queues have been used to investigate issues such as stock allocation (Véricourt et al., 2002), production scheduling (Zhao et al., 2008), dynamic pricing (Gayon et al., 2009b) and multi-echelon coordination (Veatch and Wein, 1994). A few make-to-stock papers include product returns (see e.g. Heyman (1977), Gayon and Dallery (2006)). However, none of them investigates the use of ARI. Our modeling of imperfect ARI is close to the modeling of imperfect Advance Demand Information (ADI) introduced by Gayon et al. (2009a). In the latter paper, the customer announces his intention to buy a product but the actual ordering takes place after a stochastic demand leadtime, with a cancellation probability. In this paper, we assume that the customer announces his intention to return a product where the actual return occurs after a stochastic return leadtime, with a return cancellation probability. ADI and ARI have opposite impacts on production control. For ADI, production is planned when there are many pending orders. For ARI, production is not planned when there are many pending returns. Because of the increasing use of ADI, we also pay some attention to the combined use of ARI and ADI.

The rest of the paper is setup as follows. First, we describe the situation that we study as well as the objective function to be optimized. Next, we derive the optimal production policy for lost sales situations for an infinite horizon. Via numerical experiments we determine the sets of parameter values for which ARI may be useful. Next we show that the model developed for the lost sales situations can be amended to deal with backlog situations. We also derive the optimal production policy when both ARI and ADI are used. Then we explain how our model can be used for other applications than product returns. Finally we briefly summarize our main findings and indicate some interesting extensions of the model presented here.

5.2 Problem description

In this paper we focus on situations where individual products are produced and returned. Products that are returned are as good as new, and are stored in the stock of serviceable products together with the products that the company produces new.

We consider an M/M/1 make-to-stock queue for a single item (see Figure 5.2). The company can decide at any time to produce this item. The production time is exponentially distributed with mean $1/\mu$. After having been produced, products are stored in the serviceable products inventory. Demand for the serviceable products follows a Poisson process with rate λ. For the moment being, we assume lost sales: Demand that cannot be fulfilled immediately is lost. We will also consider backorder situations (see Section 5.4).

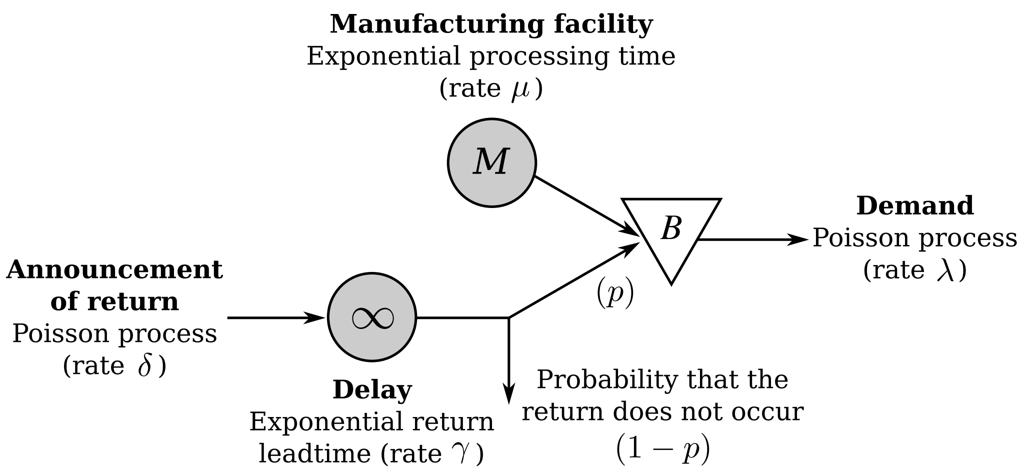


Figure 5.2: Inventory system with return flow and ARI $v^\star(x,y) = Tv^\star(x,y), \forall(x,y)$

Besides the single server production mode, the company has an alternative procurement mode where the company receives products from another source that is not under her direct control. These products can not be distinguished from the products produced by the single server. We assume that the company has some advance information on the alternative procurement process.

The alternative source considered hereafter is customers that can return products, although, as we shall indicate in Section 5.6, the following also holds for other alternative sources. Before returning a product, the customer must announce that he will return the product. The announcements occur according to a Poisson process with rate δ, independently of the demand process. However, not every announced return becomes an actual return. Reasons for this in practice include forgetting to return, not at home at the moment of planned pickup, mind change. We assume that there is a probability p that an announced return is actually returned. There is a probability $q = 1 - p$ that an announced return is cancelled. All actual returns have to be accepted and a return can not be disposed. Therefore, to guarantee the stability of the on-hand stock of serviceable products, we assume that $p\delta < \lambda$. The following notations will be useful: $\rho_1 = \lambda/(\mu + p\delta)$, $\rho_2 = p\delta/\lambda$.

We further assume that the time L that elapses between the announcement of a return and its actual receipt (or cancellation) is exponentially distributed with rateγ. This time does not depend on the number of announced returns. Note that a number of the earlier mentioned examples from practice concern situations with a predefined maximum return time. However, in practice, companies deviate from this time for all kinds of reasons, for instance to keep important customers. We make here the same approximation as many other authors, including Yuan and Cheung (1998).

Once received, a return is stored in the serviceable stock and can be sold. The state of the system can be described by $(X(t), Y(t))$ where $X(t)$ denotes the on-hand stock of new and returned products at time t, and $Y(t)$ denotes the number of returns that have been announced but still have not been received or canceled at time t.

We consider unit manufacturing cost, c^m, unit lost sale cost c^l, unit return cost c^r that only has to be paid for actual returns, and unit inventory holding cost per unit of time, h. We assume that $c^m < c^l$ in order to have an incentive to produce. The objective of the decision maker is to find a production control policy π minimizing the expected discounted cost over an infinite time horizon. The discount rate is denoted by α. The production control policy specifies, for each state of the system, when to produce. We define $v^\pi(x, y)$ as the expected total discounted cost associated with policy π, for initial state $(X(0), Y(0)) = (x, y)$.

We seek to find the optimal policy $\pi^\star$ minimizing $v^\pi(x, y)$, where we let $v^\star(x, y) = v^{\pi^\star}(x, y)$ denote the optimal value function. We restrict our analysis to stationary Markovian policies since there exists an optimal stationary Markovian policy (Puterman, 1994). In the following, we characterize the optimal policy for the case where ARI is used and for the case where ARI is ignored.

5.3 Lost sales situations

5.3.1 Optimal policy when ARI is used

When ARI is taken into account, decisions are based on both the on-hand stock of serviceable products, $X(t)$, and the number of announced returns, $Y(t)$. The situation can be modeled as a continuous-time Markov Decision Process (MDP). In order to uniformize this MDP (Lippman, 1975), we assume that the number of announced returns is bounded by M. This is not a crucial assumption since our results will hold for any$M$. We choose a uniformization rate $C = \lambda + \mu + \delta + M\gamma$. The optimal value function can be shown (Puterman, 1994) to satisfy the optimality equations

$$v^\star(x,y) = \mathcal{T}v^\star(x,y), \forall(x,y),$$

where the operator $\mathcal{T}$ is a contraction mapping defined as

$$\mathcal{T}v(x,y) = \frac{1}{C+\alpha}\left[\begin{array}{l} hx + \mu T_0 v(x,y) + \lambda T_1 v(x,y) \\ +\delta T_2 v(x,y) + \gamma p T_3 v(x,y) \\ +\gamma(1-p)T_4 v(x,y) \end{array}\right]$$

with

$$T_0 v(x,y) = \min\{v(x,y), v(x+1,y) + c^m\}$$

$$T_1 v(x,y) = \begin{cases} v(x-1,y) & \text{if } x > 0 \\ v(x,y) + c^l & \text{if } x = 0 \end{cases}$$

$$T_2 v(x,y) = \begin{cases} v(x,y+1) & \text{if } y < M \\ v(x,y) & \text{if } y = M \end{cases}$$

$$T_3 v(x,y) = \begin{cases} y(v(x+1,y-1) + c^r) + (M-y)v(x,y) & \text{if } y > 0 \\ Mv(x,y) & \text{if } y = 0 \end{cases}$$

$$T_4 v(x,y) = \begin{cases} yv(x,y-1) + (M-y)v(x,y) & \text{if } y > 0 \\ Mv(x,y) & \text{if } y = 0 \end{cases}$$

Operator T_0 is related to the production decision. Operator T_1 is associated with the demand. Operator T_2 corresponds to the announcements of the returns. Finally, operator T_3 (resp. T_4) is related to an announcement that will (resp. will not) actually lead to a return. Considering operator T_0, we notice that the optimal production control is entirely determined by the sign of $(\Delta v^\star(x,y) + c^m)$ where $\Delta v(x,y) = v(x+1,y) - v(x,y)$. In order to characterize the optimal policy, we prove that $v^\star$ belongs to the following set U of real-valued functions.

Definition 5.3.1. If $v \in U$, then for all (x,y):

C.1 $\Delta v(x,y) \leq \Delta v(x+1,y)$.

C.2 $\Delta v(x, y) \leq \Delta v(x, y+1), \forall y < M$.

C.3 $\Delta v(x, y+1) \leq \Delta v(x+1, y), \forall y < M$.

C.4 $\Delta v(x, y) \geq -c^l$.

Lemme 5.3.1. *If $v \in U$, then $\mathcal{T}v \in U$. Moreover the optimal value function $v^\star$ belongs to U.*

The proof of Lemma 5.3.1, based on an induction argument, is given in Appendix C.1 at the end of this paper. As $v^\star$ satisfies C.1, we can define the threshold $S(y) = \min\{x | \Delta v^\star(x, y) + c^m > 0\}$ such that $\Delta v^\star(x, y) + c^m \leq 0$ (i.e. it is optimal to produce) if and only if x is below this threshold. Conditions C.2 and C.3 imply a slope of $S(y)$ between 0 and 1 ($S(y) - 1 \leq S(y+1) \leq S(y)$), i.e. an additional announced return leads to at most a one unit decrease of the threshold.

Theorem 5.3.2. *There exists a switching curve $S(y)$ such that it is optimal to produce if and only if $x < S(y)$. Moreover, the switching curve has the following property:*

$$S(y) - 1 \leq S(y+1) \leq S(y)$$

Figure 5.3 illustrates the optimal policy when using ARI with respect to when to produce and when not to produce, for a given set of parameter values.

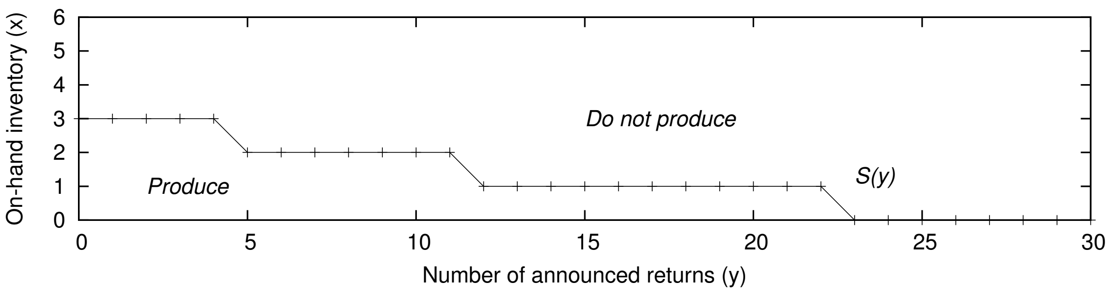


Figure 5.3: The structure of the optimal policy when ($\lambda = 1, \mu = 1, \delta = 0.25, \gamma = 0.75, c^r = 0, c^m = 0, h = 1, c^l = 10, p = 0.25$).

5.3.2 Optimal policy when ARI is ignored

When ARI is ignored, the decision maker does not take into account the announced returns to make production decisions. Then the state of the system can be described by the single variable $X(t)$. The physical returns to the stock occur according to a Poisson process with rate $p\delta$. This is due to the property that the output process of an $M/M/\infty$ queue is a Poisson process with rate equal to the arrival rate (Gross and Harris, 1998). Hence, when ARI is ignored, the system behaves like an $M/M/1$ make-to-stock queue with independent Poisson demand and returns (see Figure 5.4) where the parameter γ can be omitted.

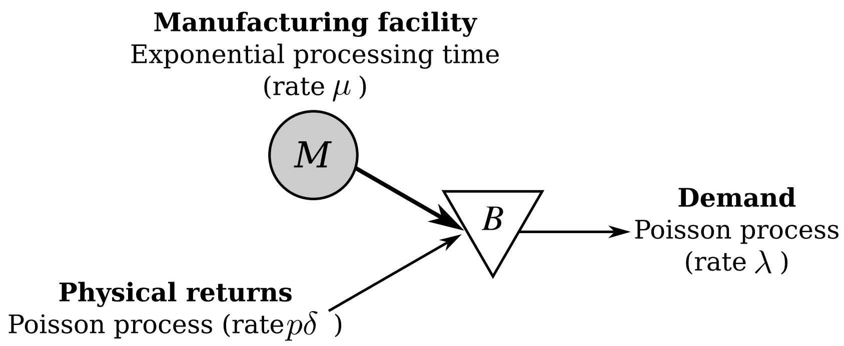


Figure 5.4: Inventory system with return flow and without ARI

For this system, Zerhouni et al. (2010) show that the optimal policy is a base-stock policy with a single parameter $S^\star$ such that it is optimal to produce if and only if $x < S^\star$. When a base-stock policy is applied, the dynamics of the system is rather simple. For a given base-stock level S, the on-hand stock $X(t)$ evolves as a continuous-time Markov chain (see Figure 5.5).

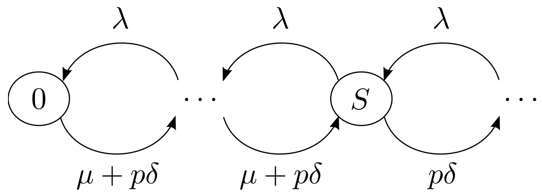


Figure 5.5: Markov chain for the system without ARI

It is straightforward to derive the steady-state probability $\pi_x(S)$ to be in state x when the base-stock level is S:

$$\pi_0(S) = \rho_1^S \left(\tfrac{1-\rho_1^{S+1}}{1-\rho_1} + \tfrac{\rho_2}{1-\rho_2} \right)^{-1},$$
$$\pi_x(S) = \begin{cases} \rho_1^{-x}\pi_0(S) & \text{if } x \leq S, \\ \rho_1^{-S}\rho_2^{x-S}\pi_0(S) & \text{if } x > S. \end{cases}$$

Then the average cost $C(S)$ with respect to a base-stock level S can be expressed as

$$\begin{aligned} C(S) &= p\delta c^r + \lambda c^l \pi_0(S) + \mu c^m \sum_{x=0}^{S-1} \pi_x(S) + h \sum_{x=1}^{+\infty} x\pi_x(S) \\ &= p\delta c^r + \lambda c^l \pi_0(S) + \mu c^m \left(\tfrac{1-\rho_1^{-S}}{1-\rho_1^{-1}} \right) \\ &\quad + h\pi_0(S)\rho_1^{-S} \left(\tfrac{\rho_1^{S+1} - \rho_1 - \rho_1 S + S}{(1-\rho_1)^2} + \tfrac{\rho_2(1+(1-\rho_2)S)}{(1-\rho_2)^2} \right). \end{aligned}$$

When $\rho_1 \leq 1$, Zerhouni et al. (2010) show that this average cost is convex. In this case, any convex optimization procedure can be used to find the optimal base-stock level. When

$\rho_1 > 1$, Zerhouni et al. (2010) show that the average cost is bounded above by

$$S_u = (\lambda - \mu)c^l/h + [\ln(\lambda/\mu)]^{-1}.$$

In this case, an exhaustive search of the optimal base-stock level on the set $\{0, \ldots, S_u\}$ can be executed.

5.3.3 Comparing the two policies

In a numerical study, we investigate the value of using ARI by comparing the results obtained for the two policies introduced in sections 3.1 and 3.2.

We focus on the average cost criterion. The average cost optimal policy can be shown to be the limit of the discounted cost optimal policy when the discount rate α goes to 0, by theorems 7.2.3 and 7.5.6 of Sennott (1999). Therefore the structures of the optimal policies are similar to the ones introduced in sections 5.3.1 and 5.3.2.

We denote by $g(ARI)$ and $g(noARI)$ the optimal average costs when using ARI and not using ARI. In order to compare the two policies, we look at the percentage cost increase for not using ARI, defined as $\Delta g = (g(noARI) - g(ARI))/g(ARI)$.

Computational procedure

To compute the optimal average costs, we use a value iteration algorithm Puterman (1994). The iteration is terminated when a six digit accuracy is achieved. To implement this algorithm, we need to truncate the state space. We repeat the value iteration algorithm with larger and larger state spaces until the cost is no longer sensitive to increasing the state space, with a six digit accuracy. With a standard PC, computing the optimal policy takes in general less than a minute. However, it also may take several hours, e.g. for large return leadtimes or when the system approaches instability of inventory (i.e. when $p\delta$ close to λ).

Experimental design

The model presented in Section 2 includes 9 parameters $(\lambda, \mu, \delta, \gamma, c^r, c^m, h, c^l, p)$. However, in this numerical study, we concentrate on varying 5 parameters$(\mu, \delta, \gamma, c^l, p)$. Without losing generality, we can set $\lambda = 1$ and $h = 1$ since this is equivalent to choosing the time and monetary unit to be used in the calculations.

In the following, we show that we can set $c^r = 0$ and $c^m = 0$ when we investigate the added value of ARI. Consider Problem A with parameter values $(\lambda, \mu, \delta, \gamma, c^r, c^m, h, c^l, p)$ and Problem B with parameter values $(\overline{\lambda}, \overline{\mu}, \overline{\delta}, \overline{\gamma}, \overline{c^r}, \overline{h}, \overline{c^m}, \overline{c^l}, \overline{p})$. Assume that the parameter values are identical for both problems except for the production cost, lost sale cost and return cost: $\overline{c^l} = c^l - c^m, \overline{c^m} = 0, \overline{c^r} = 0$.

Property 5.3.3.

i. *When ARI is used (resp. ARI is not used), the optimal production policy for problems A is also optimal for problem B.*

ii. *We have the following relation between the optimal average cost of problem A, $g(ARI)$, and the optimal average cost of problem B, $\overline{g}(ARI)$,*

$$g^A(ARI) = g^B(ARI) + \delta p(c^r - c^m) + \lambda c^m,$$
$$g^A(noARI) = g^B(noARI) + \delta p(c^r - c^m) + \lambda c^m.$$

iii. *The percentage cost increase for problem A, Δg, is smaller than the one for problem B, $\Delta \overline{g}$*

The proof is detailed in Appendix C.2. For the remaining parameters $(\mu, \delta, \gamma, c^l, p)$, we have considered the following values:

$$c^l \in \{1, 10, 100\}, \delta \in \{0, 0.25, 0.5, 0.75, 0.95\},$$
$$\gamma \in \{0.1, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2\},$$
$$\mu \in \{0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 10\}, p \in \{0, 0.25, 0.5, 0.75, 1\}.$$

We did numerical experiments for the 4800 possible combinations of the above values for $\lambda = 1$, $h = 1$, $c^r = 0$ and $c^m = 0$.

Discussion results

When does ARI make sense, i.e. when does ARI result in "significantly" smaller average cost when compared with not using ARI, and when not?

The maximum Δg over all scenarios is 4.5%, which is observed for the following combination of parameter values: $\mu = 1, \delta = 0.75, \gamma = 0.1, c^l = 100, p = 1$. Observe that, thanks to Property 5.3.3, this result remains valid for any positive return cost c^r and any combination of c^m and c^l such that $(c^l - c^m) \in \{1, 10, 100\}$.

In 91% of the examined scenarios, $\Delta g < 1\%$ and in 97% of the scenarios, $\Delta g < 2\%$. Whether ARI is useful depends on the exact combination of the parameter values. We have observed that Δg is non-monotonic with respect to all parameters. Figure 5.6 shows the influence of $1/\gamma$ (i.e. the average time between the announcement of a return and the actual receipt of the return or decline of the return by the customer) on Δg for some values of p. We observe that the relative benefit for using ARI tends to be insignificant when $1/\gamma$ is either small or large. When $1/\gamma$ is small, the explanation is simple: Returns are announced right before they arrive and ARI is useless. When $1/\gamma$ is large, returns are announced far in advance. Due to the exponential distribution choice for the return leadtime, when the expected value of the return leadtime increases, so does the variance and this makes ARI less useful. For intermediate values of $1/\gamma$, using ARI is most beneficial. We obtain similar results for the other combinations of parameter values examined in this paper.

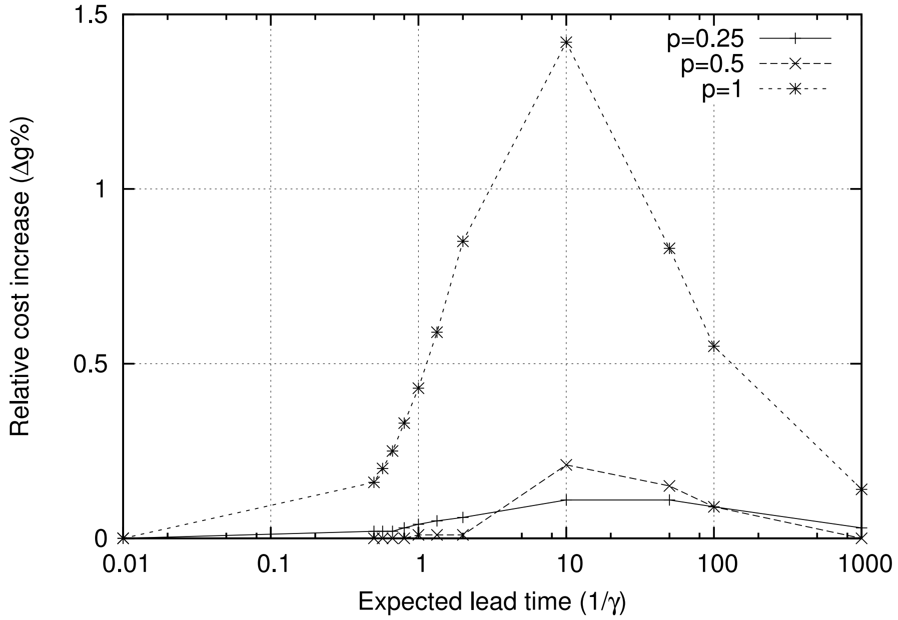


Figure 5.6: The effect of the expected return leadtime on the value of using ARI ($\lambda = 1, \mu = 1, \delta = 0.25, c^r = 0, c^m = 0, h = 1, c^l = 100$)

The previous observations hold for exponential return leadtimes but do they also pertain to other return leadtime distributions? To be able to compute the optimal policy, we need to consider leadtime distributions that are combinations of exponential distributions. Hereafter, we consider a return leadtime L distributed according to an Erlang-k distribution. Let $L = L_1 + \cdots + L_k$ where $L_1, \cdots, L_k$ are independently and exponentially distributed with rate $k\gamma$. Then the expected value of L is $1/\gamma$ and its standard deviation is $1/\sqrt{k}\gamma$. Figure 5.7 plots the effect of $1/\gamma$ for some values of k. We observe that the benefit of ARI is increasing with k. This is logical since the variance of the return leadtime is decreasing with k and the 'quality' of ARI is getting better. We also observe that the benefit of ARI remains maximal for intermediate values of $1/\gamma$. This may not be the case in general and especially for large values of k for which the standard deviation is smaller. Computing the optimal policy for large values of k is intractable due to the curse of dimensionality. When k goes to infinity, the return leadtime is converging to the constant leadtime$L = 1/\gamma$. In this case, the decision maker has, at time t, exact information on the timing of all returns in the time window $[t, t+L]$. Increasing the horizon of visibility L will necessarily decrease the average cost. Hence, the benefit of ARI should be nondecreasing with the return leadtime, when deterministic.

Figure 5.8 shows the influence of p on Δg for different values of δ. When $p\delta$ goes to λ or to 0, we observe that Δg goes to 0. When $p\delta$ goes to λ, returns are sufficient to satisfy the demand and it is no more necessary to produce. With or without ARI, the optimal policy consists in not producing all the time and ARI is again useless. When $p\delta = 0$, there are no actual returns and ARI is useless. The curves in Figure 5.8 show irregularities which

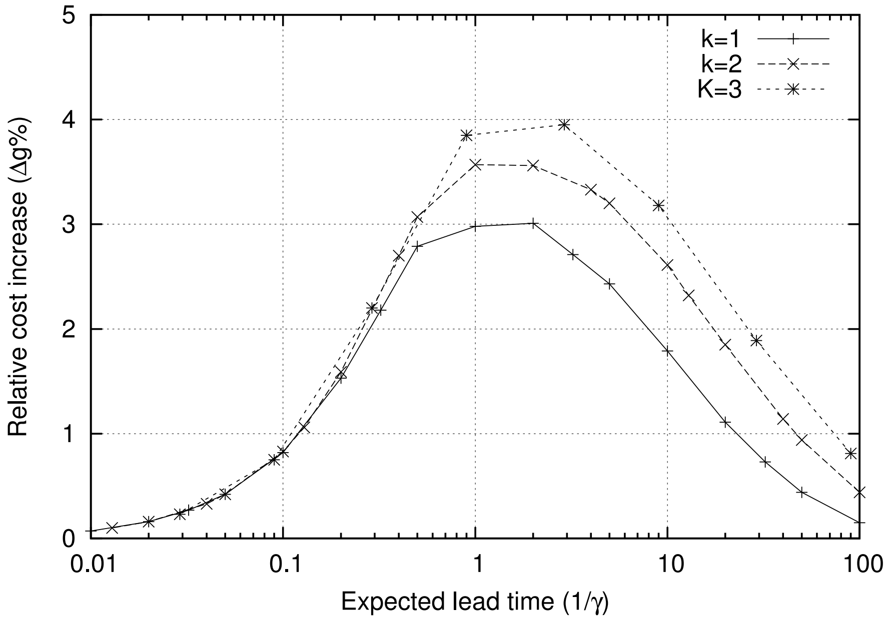


Figure 5.7: The effect of the expected return leadtime on the value of using ARI for Erlang-k distributions ($\lambda = 1, \mu = 2, \delta = 0.5, c^r = 0, c^m = 0, h = 1, c^l = 10, p = 1$).

are due to the discrete nature of the base-stock levels. To make this relation clear, we plotted Δg and the optimal base-stock level $S^\star$ for the situation without ARI (Figure 5.9). We observe that the irregularities in the curve Δg coincide with changes in the optimal base-stock level.

Now we look at the influence of γ and p on the optimal policy. Figure 5.10 (respectively Figure 5.11) plots the state-dependent base-stock level $S(y)$ as a function of y for different values of γ (respectively p). We observe in Figure 5.10 that $S(y)$ is nonincreasing with the return rate γ. The larger the return lead time, the more we should produce. Moreover, when the return lead time is very short ($\gamma = 100$), the optimal policy consists in producing when $x + y < 4$ where 4 is precisely the optimal base-stock level when not using ARI. In this case, announced returns can be considered to be already in stock. In Figure 5.11, we observe that $S(y)$ is nonincreasing with the return probability p. It seems logical that we produce less when there is a higher probability that returns arrive. When $p = 0$, there is no return at all and the base-stock level $S(y)$ is independent of y.

5.4 Backorder situations

5.4.1 Optimal policy when ARI is used

We also consider situations with backorders (with linear backorder cost $b$per unit of time) instead of lost sales. The model formulation is slightly different but leads to similar results. In this case, on-hand stock $X(t)$ is replaced by net stock $\tilde{X}(t)$ with $\tilde{X}^+(t) = \max\left[0, \tilde{X}(t)\right]$,

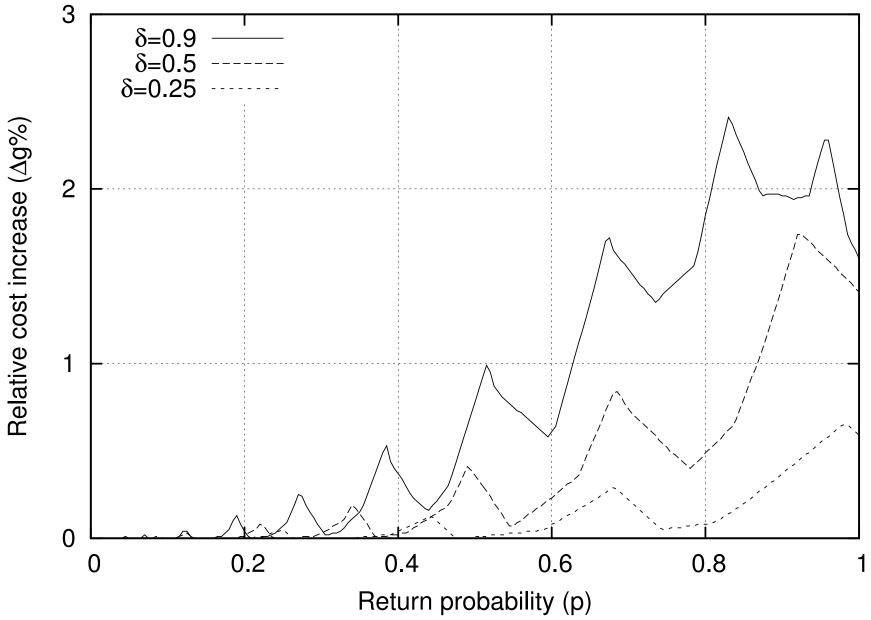


Figure 5.8: The effect of the expected return leadtime on the value of using ARI ($\lambda = 1, \mu = 1, \gamma = 0.8, c^r = 0, c^m = 0, h = 1, c^l = 100$).

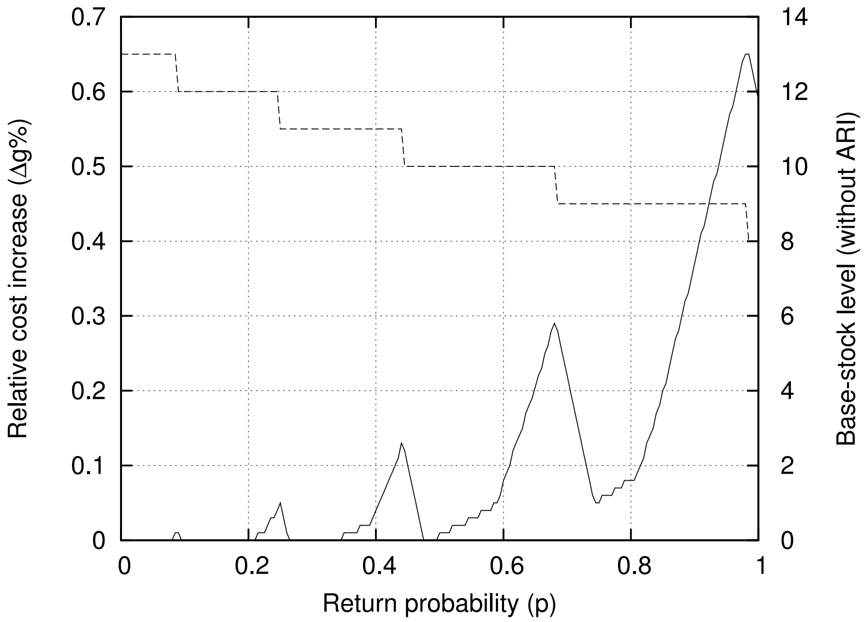


Figure 5.9: The effect of the expected return leadtime on the value of using ARI ($\lambda = 1, \mu = 1, \delta = 0.25, \gamma = 0.8, c^r = 0, c^m = 0, h = 1, c^l = 100$).

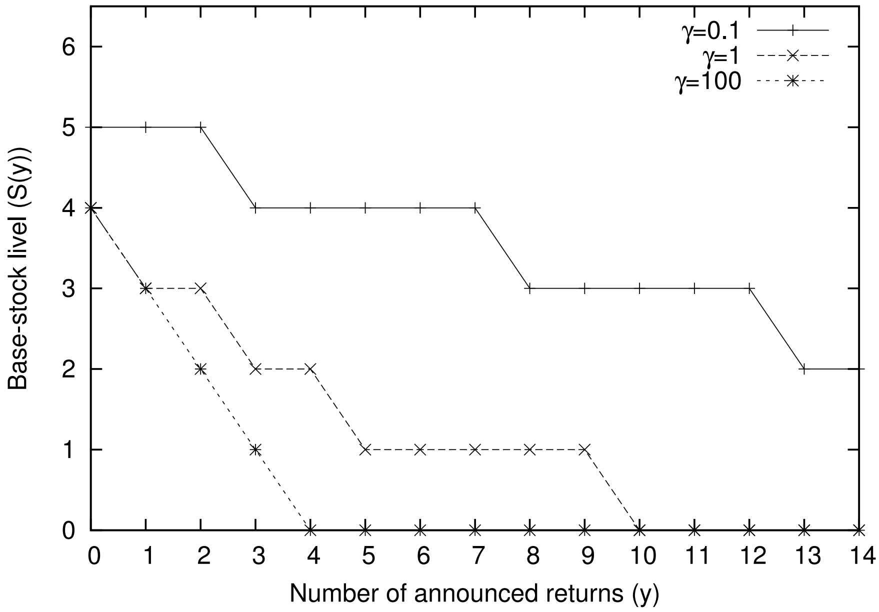


Figure 5.10: The effect of the return rate γ on the optimal policy ($\lambda = 1, \mu = 2, \delta = 0.5, c^r = 0, c^m = 0, h = 1, c^l = 100, p = 1$).

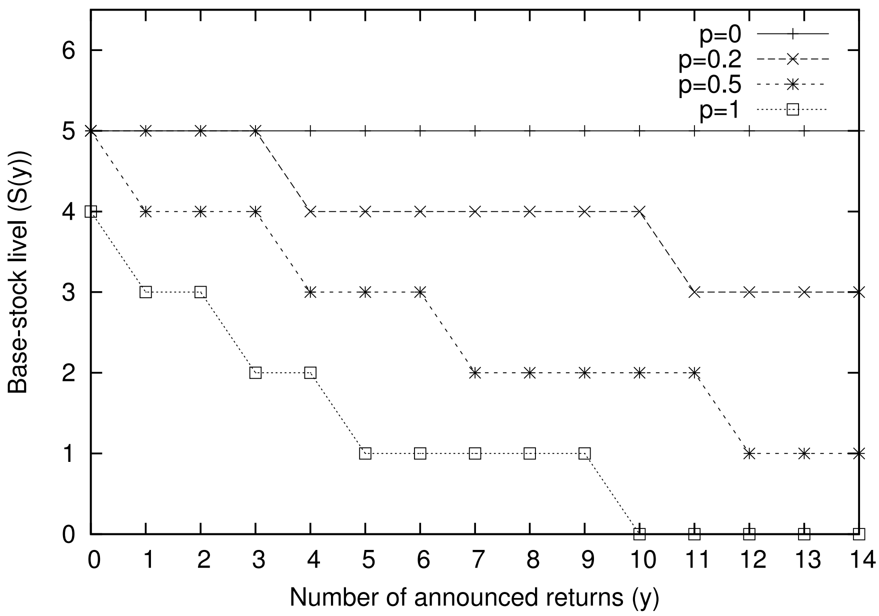


Figure 5.11: The effect of the return probability p on the optimal policy ($\lambda = 1, \mu = 2, \delta = 0.5, \gamma = 1, c^r = 0, c^m = 0, h = 1, c^l = 100$).

the on-hand stock of serviceable products, and $\tilde{X}^-(t) = \max\left[0, -\tilde{X}(t)\right]$, the number of backorders. For the discounted cost problem, the optimal value function $\widetilde{v}^\star$ satisfies the optimality equation:

$$\widetilde{v}^\star(x,y) = \widetilde{\mathcal{T}}\widetilde{v}^\star(x,y), \forall(x,y),$$

where the operator $\widetilde{\mathcal{T}}$ is defined as

$$\widetilde{\mathcal{T}}v(x,y) = \frac{1}{C+\alpha}\left[\begin{array}{l} hx^+ + bx^- + \mu T_0 v(x,y) + \lambda \widetilde{T_1} v(x,y) + \delta T_2 v(x,y) \\ +\gamma p T_3 v(x,y) + \gamma(1-p) T_4 v(x,y) \end{array}\right].$$

Operators T_0, T_2, T_3, T_4, and C are defined as in Section 5.3 and operator $\widetilde{T_1}$ is defined as

$$\widetilde{T_1}v(x,y) = v(x-1,y)$$

We define a set of functions $\widetilde{U}$ satisfying all conditions of U except Condition C.4. Then the proof is similar to the one of Lemma 5.3.1 and the optimal value function can be shown to satisfy conditions C.1, C.2 and C.3. We conclude that Theorem 5.3.2 can be extended to the case with backorders.

Theorem 5.4.1. *There exists a switching curve $\widetilde{S}(y)$ such that it is optimal to produce if and only if $x < \widetilde{S}(y)$. Moreover, the switching curve has the following property:*

$$\widetilde{S}(y) - 1 \leq \widetilde{S}(y+1) \leq \widetilde{S}(y).$$

Note that the switching curve $\widetilde{S}(y)$ can take negative values in the backorder case. This occurs when the number of announced returns is large. In this case, it can be optimal not to produce, even if there are orders waiting to be satisfied.

5.4.2 Optimal policy when ARI is ignored

When ARI is ignored, the optimal policy is again a base-stock policy and the optimal base-stock level can be computed explicitly (see Chapter 3). If we let $W = S - X$, the average cost can be written as

$$\tilde{C}(S) = hE(X)^+ + bE(X)^- = hE(S-W)^+ + bE(S-W)^-.$$

It is precisely the objective function of a newsboy problem with stochastic demand W. Let F_W denote the probability distribution function (p.d.f.) of W. The optimal base-stock level $\tilde{S}^\star$ is then $\tilde{S}^\star = \min\{z : F_W(z) > h/(h+b)\}$.

As the Markov chain $\tilde{X}(t)$ has a simple birth-death structure, the probability mass function (pmf) of W can be easily derived:

$$F_W(z) = \begin{cases} \frac{(1-\rho_1)\rho_2^{-z}}{1-\rho_1\rho_2} & \text{if } z \leq 0, \\ 1 - \frac{(1-\rho_2)\rho_1^{z+1}}{1-\rho_1\rho_2} & \text{if } z \geq 0. \end{cases}$$

We finally obtain an explicit formula for the optimal base-stock level. When $F_W(0) \geq b/(h+b)$, the optimal base-stock level is nonpositive and is given by

$$\tilde{S}^\star = \left\lceil \frac{\ln\left(\frac{1-\rho_1}{1-\rho_1\rho_2}\frac{b+h}{b}\right)}{\ln(\rho_2)} \right\rceil .$$

When $F_W(0) \leq b/(h+b)$, the optimal base-stock level is nonnegative and is given by

$$\tilde{S}^\star = \left\lfloor \frac{\ln\left(\frac{1-\rho_1\rho_2}{1-\rho_2}\frac{h}{h+b}\right)}{\ln(\rho_1)} \right\rfloor .$$

5.4.3 Comparing the two policies

Similarly to the lost sales case, we can compute the percentage cost increase for not using ARI in backorder situations. We keep the same system parameter values for $(\lambda, \mu, \delta, \gamma, c^r, c^m, h, p)$ and we vary the backorder cost b in $\{1, 10, 100\}$. Interestingly, the percentage cost increase Δg is higher in backorder situations and attains 23% for the following combination of parameter values: $\mu = 10, \delta = 0.5, \gamma = 1, b = 1, p = 1$. For this instance, the optimal policy without ARI works in a make-to-order fashion (produce if and only if the net stock is negative). Such a situation does not occur with lost sales. In 78% of the examined scenarios, $\Delta g < 1\%$ and in 96% of the scenarios, $\Delta g < 5\%$. The other insights discussed in Section 5.3.3 for lost sales situations pertain to the case with backorders.

5.5 Combining ARI and ADI

So far, only returns where announced in advance. In this section, we extend our analysis to include Advance Demand Information (ADI). Following the framework of Gayon et al. (2009a), we assume now that customers place orders according to a Poisson process with rate λ. After a demand lead time exponentially distributed with rate ν, an order becomes due with probability a or is canceled with probability $(1-a)$. The state of the system can now be described by a triplet $(X(t), Y(t), Z(t))$ where $Z(t)$ is the number of orders that have been announced but that are not due yet (or canceled) at time t. We assume that the number of orders $Z(t)$ is bounded by N. In what follows, we focus on lost sales situations but it can be extended to backorder situations.

We choose an uniformization rate $D = \lambda + \mu + \delta + M\gamma + N\nu$. The optimal value function $\bar{v}^\star$ satisfies the optimality equations $\bar{v}^\star = \bar{\mathcal{T}}\bar{v}^\star$ where $\bar{\mathcal{T}}$ is defined as

$$\bar{\mathcal{T}}\bar{v}(x,y,z) = \frac{1}{D+\alpha}\left[\begin{array}{l} hx + \mu T_0 v(x,y,z) + \delta T_2 v(x,y,z) + \gamma p T_3 v(x,y,z) \\ +\gamma(1-p)T_4 v(x,y,z) + \lambda \bar{T}_1 v(x,y,z) \\ +\nu a T_5 v(x,y,z) + \nu(1-a)T_6 v(x,y,z) \end{array}\right].$$

Operators T_0, T_2, T_3, T_4 are defined as in Section 5.3 and operator $\bar{T}_1$, T_5, T_6 are defined as

$$\bar{T}_1 v(x,y,z) = \begin{cases} v(x,y,z+1) & \text{if } z < N, \\ v(x,y,z) + c^l & \text{if } z = N, \end{cases}$$

$$T_5 v(x,y,z) = \begin{cases} zv(x-1,y,z-1) + (N-z)v(x,y,z) & \text{if } z > 0 \text{ and } x > 0, \\ z[v(x,y,z-1) + c^l] + (N-z)v(x,y,z) & \text{if } z > 0 \text{ and } x = 0, \\ Nv(x,y,z) & \text{if } z = 0, \end{cases}$$

$$T_6 v(x,y,z) = \begin{cases} zv(x,y,z-1) + (N-z)v(x,y,z) & \text{if } z > 0, \\ Nv(x,y,z) & \text{if } z = 0. \end{cases}$$

Operator $\bar{T}_1$ is associated to the announcement of customer orders. Operator T_5 (resp. T_6) is related to actual demands (resp. cancellations). The optimal production control is entirely determined by the sign of $(\Delta\bar{v}^\star(x,y,z) + c^m)$ where $\Delta v(x,y,z) = v(x+1,y,z) - v(x,y,z)$. In order to characterize the optimal policy, we prove that $\bar{v}^\star$belongs to the following set V of real-valued functions.

Definition 5.5.1. If $v \in V$, then for all (x,y,z):

C.1 $\Delta v(x,y,z) \leq \Delta v(x+1,y,z)$.

C.2 $\Delta v(x,y,z) \leq \Delta v(x,y+1,z), \forall y < M$.

C.3 $\Delta v(x,y+1,z) \leq \Delta v(x+1,y,z), \forall y < M$.

C.4 $\Delta v(x,y,z) \geq -c^l$.

C.5 $\Delta v(x,y,z) \geq \Delta v(x,y,z+1), \forall z < N$.

C.6 $\Delta v(x,y,z) \leq \Delta v(x+1,y,z+1), \forall z < N$.

We show in Appendix C.3 that the optimal value function $\bar{v}^\star$ belongs to V by combining the proof when only ARI is used (Lemma 5.3.1, Section 5.3) and the proof when only ADI is used (Lemma 1 in Gayon et al. (2009a)). It implies that the optimal policy has the following characteristics.

Theorem 5.5.1. *There exists a switching curve $\bar{S}(y,z)$ such that it is optimal to produce if and only if $x < \bar{S}(y,z)$. Moreover, the switching curve has the following properties:*

$$\bar{S}(y,z) - 1 \leq \bar{S}(y+1,z) \leq \bar{S}(y,z),$$
$$\bar{S}(y,z) \leq \bar{S}(y,z+1) \leq \bar{S}(y,z) + 1.$$

The first (respectively second) property of the switching curve in Theorem 5.5.1 corresponds to the property for the switching curve when only ARI (respectively ADI) is used. We can easily adapt Theorem 5.5.1 to backorder situations. It suffices to remove condition C.4 in the definition of set V.

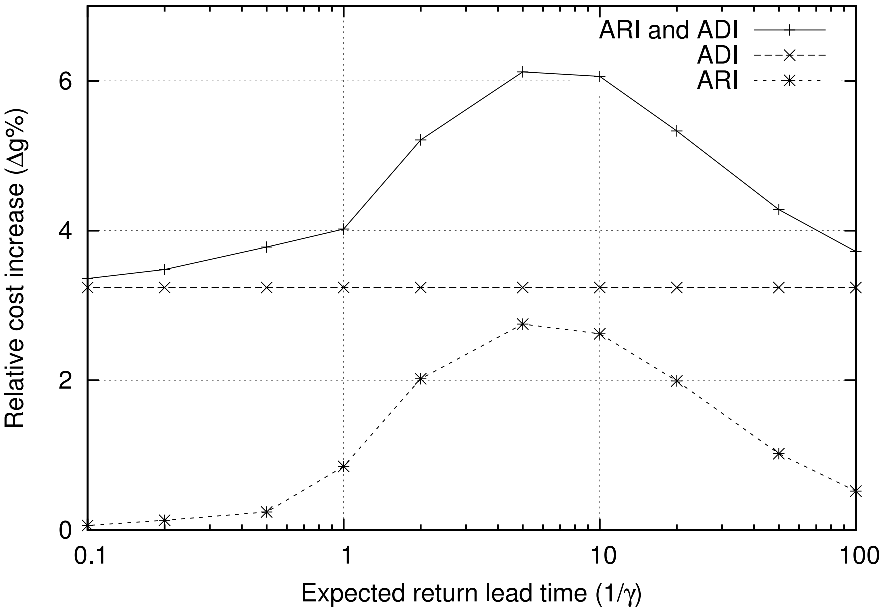
In Figure 5.12, we show illustrative results depicting the impact of ARI alone, ADI alone and joint ARI and ADI. The results indicate that the benefits of ARI and ADI are complementary. The benefit due to ADI is more significant for the instances we have tested.

5.6 Other applications

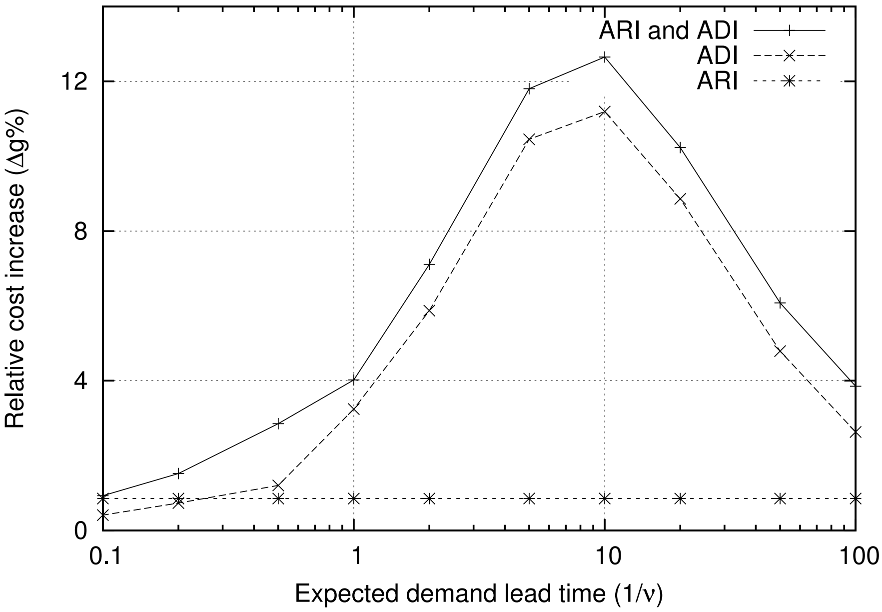
In this paper, we focused on the value of ARI for situations with returns. There are many more situations where using advance supply information (ASI) may be profitable, which are not related to return flows. The model presented in this paper can be applied to some other situations, after minor changes.

One other application concerns production planning in situations where, apart from the primary process for generating a certain product P1, the production of other products P2, P3,... via other processes may result in P1 as well, as an undesired co- or by-product (production with a variable quality yield). Such a situation can among others be found in the process industries, where complete batches from one process can not have the desired quality from the point of view of this process, but can have the correct quality for another application for which normally a separate second production process is started on another facility, where customers buy complete batches one by one. In this case an announcement corresponds with the announcement of the startup of a batch for product P2, P3,... where δ indicates the arrival process of the above announcements. Note that in this case δ may be greater than λ. In this case, L denotes the time between the above announcements and the moment that the results of the related quality measurements become available.

The presented model can also be useful for companies that don't produce but buy from an external supplier with limited capacity. Some of these companies simultaneously try to buy individual (un)used products via e.g. the Internet, spot markets, auctions, where it is uncertain whether or not a company will receive the desired products because bids by others may be higher. Examples are airlines, transportation companies with big truck fleets, which follow the above strategy for expensive parts like engines, requiring a negligible effort to make them as good as new. In this case, δ indicates the arrival process



(a) ($\nu = 1$)



(b) ($\gamma = 1$)

Figure 5.12: The effect of ARI versus ADI ($\lambda = 1, \mu = 1, \delta = 0.5, c^r = 0, c^m = 0, h = 1, c^l = 100$).

of interesting announcements for which the company always bids (e.g. announcements by other companies in the same sector that decide to replace part of their fleet earlier than expected and due to this are confronted with obsolete stocks that they want to sell) whereas L denotes the time between the announcement / bid and the result of the bid.

5.7 Summary and conclusions

In this paper, we have examined the potential value of using imperfect advance supply information from a number of autonomous external sources for a company supplying one item, where the company also has one own production facility under complete control. We focused on the information that becomes available after products have been sold or lease contracts have started.

For both lost sales and backorder situations, we have characterized the optimal policy with and without ARI. In case of lost sales, it was shown that for the many sets of parameter values studied, using ARI as indicated in this paper can result in a cost reduction of 5% at most, but in 91% of the examined scenarios, the cost reduction was less than 1% and in 97% of the scenarios less than 2%. Although this may not seem much, as always we should compare this reduction with the net profit of a company, due to which the gain may be considerable. In general, our research shows that ARI, as used in this paper, seems to be most advantageous in situations where return times are not very short or very long, where the probability that an announced return becomes an actual return is also not very high or very low. When backorders are allowed, it was shown that the cost reduction can be higher, up to 23%. We have extended our analysis when both ARI and ADI are available. We also have indicated that our model can not only be useful in situations with returns from customers, but also in many other situations, like in situations with co- and by-products as well as for fleet owners.

Our model can be extended in several ways. One extension (for both lost sales and backorder situations) is to include an admission control for returns. When a customer announces the intended return of a product, the decision maker decides whether or not to accept the potential return. One reason for rejecting a return is e.g. to avoid excess inventory. Then, the optimal policy is expected to consist of two switching curves, $R(y)$ and $S(y)$ such that it is optimal to accept a return (resp. to produce) if and only if the on-hand stock of serviceable products is below $R(y)$ (resp. $S(y)$). It is also possible to consider several types of returns and to investigate how to coordinate production and admission of the different return flows. Another extension would be to study the effect of the price offered for returns.

Chapter 6

Influence of system parameters on the optimal policy in a class of multidimensional queueing control problem

With a theoretical point of view, this chapter provides a general framework to study the influence of the system parameters on the optimal policy of this system. First we provide a general modeling framework to investigate dynamic optimization problem encountered in the previous chapters. This framework employs event operators to describe the optimal value function. We then decompose a generic model with two types of operators: the choice operator and the translation operator. We provide some properties on these two types of operators that imply monotonic effect of the system input parameters on the optimal policy. More specifically we explain how the monotone behavior of the optimal policy is linked with supermodularity properties and how these properties can be proved by induction in a general setting. Then, we provide sufficient conditions to have the two studied operators propagating this properties. Two examples are given to illustrate our study. Finally, we extend our results to problems with state dependent service rates.

6.1 Introduction

The present work deals with the comparison of optimal policies of Markov Decision Processes (MDP) which differ only in their input parameter values (probabilities of transition, costs,...). It is well established that in a number of queueing control problems, there is an optimal policy that can be described by thresholds, switching curves, or surfaces (see e.g. Sections 3.4, 4.4, and 5.3.1). The main contribution of our work is to study the effect of varying simultaneously several parameters on the optimal policy in a multidimensional queueing control problem.

In what follows, we present two examples to illustrate (1) the effect of varying one system parameter in a problem where the state of the system can be described by a combination of parameters x_1 and x_2 (multi-dimensional problem), and (2) the effect of varying several parameters in a problem where the state of the system can be described by one parameter x (mono-dimensional problem). To the best of our knowledge, none of these two situations have been investigated in the literature, from a theoretical perspective. These examples are used throughout this chapter to illustrate our results.

Example 1: We first present the make-to-stock tandem queue model (Veatch and Wein, 1992, 1994) in order to illustrate the effect of one system parameter variation on the optimal policy in a multidimensional queueing control problem. The make-to-stock tandem queue is a 2-stage production/inventory system in series which satisfies end-customer demand (see Figure 6.1). The servers M_1 and M_2 produce items one by one with exponentially distributed processing times, with rates μ_1 and μ_2. Produced items at server i are stocked in a buffer B_i. The end buffer B_2 sees customer demands arriving according to a Poisson process with rate λ. We assume that backorders and preemption are allowed. The state of the system is described by (x_1, x_2) with x_1 the current number of work-in-process products in buffer B_1 and x_2 the current number of serviceable products in buffer B_2 minus the current number of backlogged demand. The system incurs a holding cost h_i per unit of time and unit of product in buffer B_i and a backorder cost b per unit of time and unit of waiting demand. The objective is to minimize the expected discounted/average cost over an infinite horizon. Veatch and Wein (1992) prove that the optimal policy exists and is a state dependent base stock policy defined by two switching curves $s_1(x_2)$ and $s_2(x_1)$.

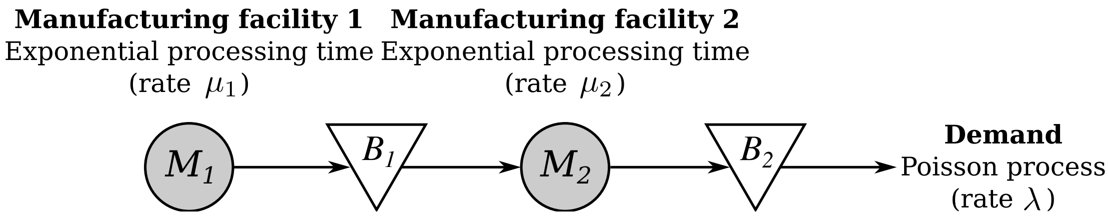


Figure 6.1: Tandem make-to-stock queue model.

Figure 6.2 (resp. 6.3) shows the influence of the demand rate λ (resp. server rate μ_2) on the switching curves. On these examples, we observe that λ has a monotonic effect on the switching curves and μ_2 has a non-monotonic effect on the switching curve s_i. In this paper we first address the following question, can we predict the effect of a parameter variation on the optimal policy for a multidimensional system?

Example 2: In order to illustrate the effect of joint variations of several system parameters on the optimal policy of a monodimensional queueing control problem we present the admission control model of Stidham (1985). The main assumptions on this system are following. The system receives 3 different types of customers, where a class$-i$ customers

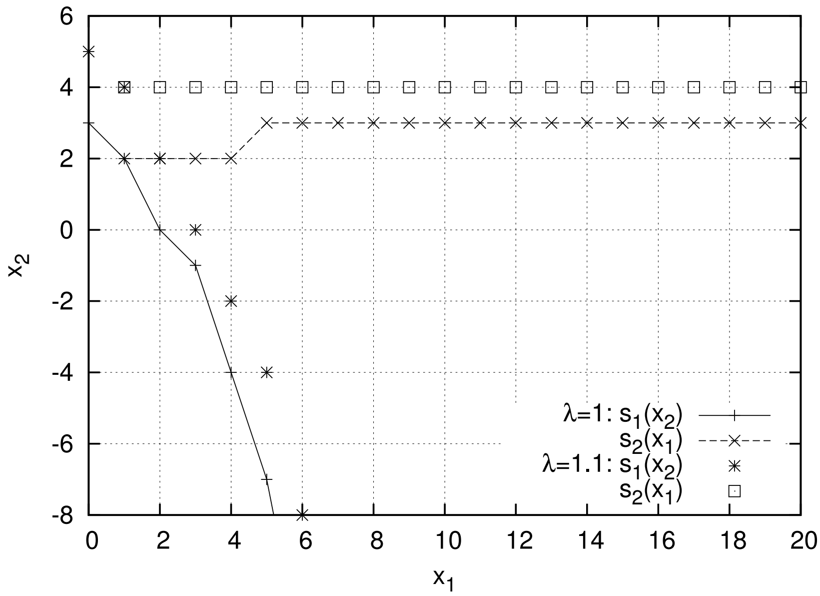


Figure 6.2: Effect of λ on the optimal policy ($\mu_1 = 2$, $\mu_2 = 1.3$, $h_1 = 1$, $h_2 = 3$, $b = 2$).

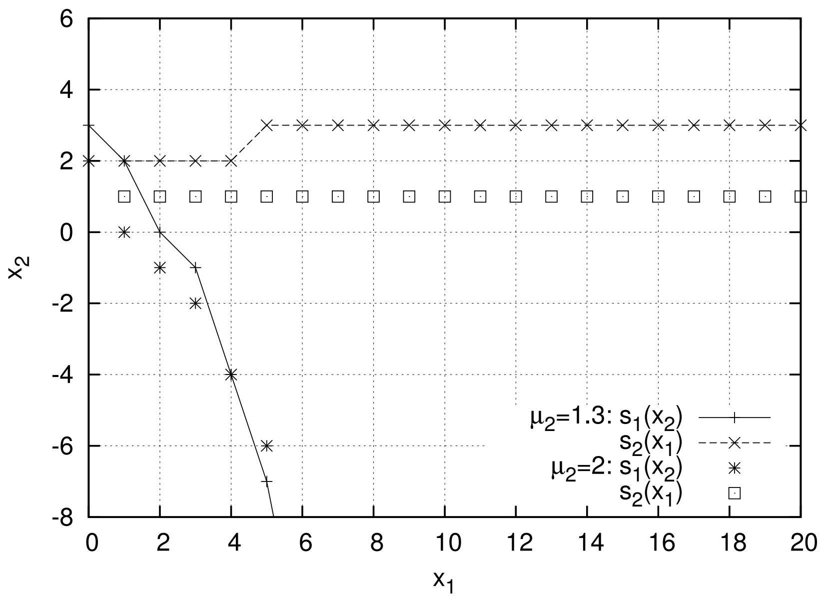


Figure 6.3: Effect of μ_2 on the optimal policy ($\lambda = 1$, $\mu_1 = 2$, $h_1 = 1$, $h_2 = 3$, $b = 2$).

arrives at the system according to a Poisson process with rate λ_i and require an exponential service time with rate μ. The server produces item one by one and the preemption is allowed. The state is the number of customers in the system $x \in \mathbb{N}$. The waiting cost is h per client per unit of time. At each arrival, the decision maker either accepts the incoming customer or rejects her with a cost R_i in order to minimize the discounted/average expected cost over an infinite horizon. Stidham (1985) proves that the optimal policy is a threshold policy where a class-i customer is accepted if and only if $x < t_i$.

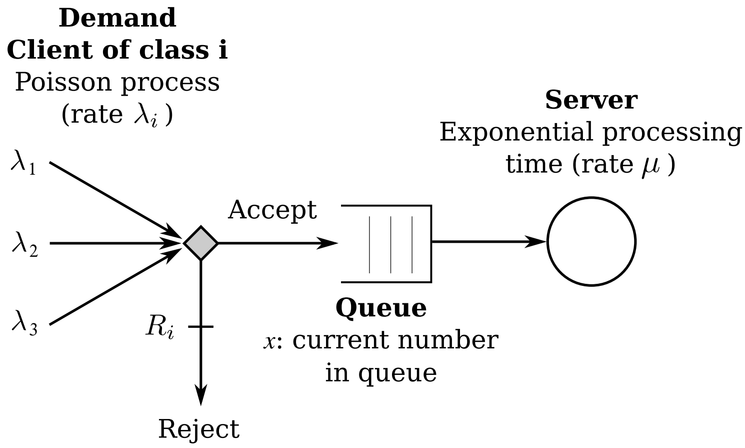


Figure 6.4: Admission control model.

Table 6.1 presents optimal threshold values as a function of arrival rates. In all presented instances the sum of arrival rates is constant, but compared with Instance 1, the thresholds of Instance 2 increase, those of Instance 3 decrease and those of Instance 4 have a non-monotonic behavior. In this paper, we will try to answer the following question. Can we predict such variations?

Instance	λ_1	λ_2	λ_3	t_1	t_2	t_3
1	0.6	0.6	0.6	9	3	1
2	0.1	0.7	1	15	6	1
3	1	0.7	0.1	6	2	0
4	0.1	1.6	0.1	13	3	0

Table 6.1: Optimal thresholds in function of arrival rates, with $\mu = 1$, $h = 1$, $R_1 = 30$, $R_2 = 20$, and $R_3 = 10$.

The rest of the paper is set up as follows. Section 6.2 presents a literature review about MDP. Section 6.3 presents the class of MDP investigated in this paper. Supermodularity properties used to predict the behavior and the structure of the optimal policies are presented in Section 6.4. Section 6.5 presents our main theorems. Section 6.6 considers the problem of joint variations of several system parameters. Finally, Section 6.7 extends the investigated class of MDP.

6.2 Literature review

In a number of queueing control problems, the structure of the optimal policy can be described by thresholds, switching curves, or hyperplanes. Several papers address the problem of finding the structure of the optimal policy for a general queueing control problem. For instance Veatch and Wein (1992) study a class of queueing network control problems and apply their work on tandem queue, assembly and routing systems. They find a similar result to Weber and Stidham (1987) about the multimodularity property in queueing networks. In a more general context, Smith and McCardle (2002) study the closed convex cone properties including monotonicity, convexity and supermodularity and derive structural properties for optimal policies. Koole (1998) presents the framework of Event Based Dynamic Programming (EBDP). He studies a class of MDPs that can be modeled by a set of events which do not occur at the same time. He proposes a general approach to study the optimal policy structure in this class of problems. Note that most of queueing control problems can be easily modeled by a set of events (arrival, departure, etc. . .). The EBDP approach is used in many papers, e.g. Koole (2004) and Zhuang and Li (2012) who extend results on multimodularity in queueing system, and Morton (2006) who uses EBDP in revenue management context. Recent syntheses and surveys about EBDP include Koole (2006) and Zhuang and Li (2012).

The first order sensitivity analysis problem consists in predicting the behavior of the optimal cost for a perturbation on the system parameter values. Müller (1997) addresses this problem for a general MDP formulation and applies his work on an inventory problem and a stopping problem. In an EBDP context, but only for perturbation on rate of uncontrolled event, Koole (2006) studies this problem and extends it to the convexity of cost in direction of the perturbations. Other papers, like Ku and Jordan (1997); Aktaran-Kalaycı and Ayhan (2009), perform a first order sensitivity analysis on specific examples.

The problem of second order sensitivity analysis is less studied. It consists of predicting the behavior of the optimal policy when the parameters of the system are perturbed. Many papers (see e.g. Gans and Savin (2007) or Chapter 5) present numerical experiments to observe the behavior of the cost and/or the optimal policy in function of parameters values. As far as we know, only Çil et al. (2009) study this problem theoretically in mono-dimensional model context. The authors propose a framework to study the effect of system parameters on the optimal policy in a class of mono-dimensional queueing control problems. They performed the analysis by varying the system parameters one by one. They applied their method on an admission control problem. Their results have been applied by Zerhouni et al. (2010) in an M/M/1 make-to-stock queue with product returns from customers and in Benjaafar et al. (2010) in a M/M/1 make-to-stock queue with customer impatience.

In this paper we extend the results of Çil et al. (2009) in two directions, (1) to a multi-dimensional problem, (2) to a joint perturbation on several system parameters. To the best of our knowledge, this problem is addressed for the first time.

6.3 Class of investigated MDPs

Consider a MDP where the state of the system is a m-dimensional vector $\boldsymbol{s}$. The MDP parameter values (transition probabilities and costs) can be summarized in a n-dimensional vector $\boldsymbol{p}$. Let $\mathbf{x} = (\boldsymbol{s}, \boldsymbol{p})$ be the $(m+n)$-dimensional vector. Our objective function is to find the set of decisions in function of all the states of the system minimizing the discounted cost over an infinite horizon. Note that a set of decisions as a function of the states of the system is called a policy.

In the rest of the paper, we are mainly interested in queueing control problems. The state $\boldsymbol{s}$ will be referred to as the queueing state while $\mathbf{x} = (\boldsymbol{s}, \boldsymbol{p})$ will be referred to as the state of the system (or simply the state). The state space of the system is denoted $\mathbb{X} \subset \mathbb{Z}^m \times \mathbb{R}^n$. The set of queues state translation (resp. system parameters perturbation) is denoted $\mathbb{A} \subset \mathbb{Z}^m \times \{0\}^n$ (resp. $\mathbb{E} \subset \{0\}^m \times \{\mathbb{R}\}^n$).

$$\begin{array}{rlccccccccl} \text{State of the system:} & \mathbf{x} = (& x_1, & \dots, & x_m & , & x_{m+1}, & \dots, & x_{m+n} &) \in \mathbb{X} \\ \text{Queues state translation:} & \mathbf{a} = (& a_1, & \dots, & a_m & , & 0, & \dots, & 0 &) \in \mathbb{A} \\ \text{System parameters perturbation:} & \boldsymbol{\epsilon} = (& 0, & \dots, & 0 & , & \epsilon_1, & \dots, & \epsilon_n &) \in \mathbb{E} \end{array}$$

Example 1: For the tandem queue problem we have $\mathbf{x} = (x_1, x_2, \mu_1, \mu_2, \lambda, h_1, h_2, b) \in \mathbb{X}$ with $\mathbb{X} = \mathbb{N} \times \mathbb{Z} \times \{\mu_1\} \times \{\mu_2\} \times \{\lambda\} \times \{h_1\} \times \{h_2\} \times \{b\}$,

Example 2: For the admission control problem we have $\mathbf{x} = (x, \mu, \lambda_1, \lambda_2, \lambda_3, h, R_1, R_2, R_3) \in \mathbb{X}$ with $\mathbb{X} = \mathbb{N} \times \{\mu\} \times \{\lambda_1\} \times \{\lambda_2\} \times \{\lambda_3\} \times \{h\} \times \{R_1\} \times \{R_2\} \times \{R_3\}$,

Koole (1998) establishes that for a general class of queueing control problems the optimality equation can be expressed in a standard form. We focus on this class of MDPs where the optimality equation can be written

$$v^\star = \mathcal{T} v^\star, \quad \text{with} \quad \mathcal{T} v = C + \sum_{i=1}^{l} p_i A_i v + p_0 v,$$

where v is a generic real valued function on $\mathbf{x} \in \mathbb{X}$, A_i is an event operator on v, p_i is the transition probability for the i-th event, τ is the discount rate such that $\tau > 0$ and $\tau + \sum_{i=0}^{l} p_i = 1$, and $C(\mathbf{x})$ is a cost function. The term $p_0 v$ is called the fictitious event (Çil et al., 2009), because with a probability p_0, no event happens. This term is used in the following to compare systems with different service rates: if the probability that an event occurs increases of ϵ, then the probability that the fictitious event occurs decreases by ϵ.

With $(\mathbf{a}, \mathbf{b}) \in \mathbb{A}^2$ and $(c, r) \in \mathbb{R}^2$ we define

- the translation operator

$$T^t_{\mathbf{a}} v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{a}) & \text{if } \mathbf{x}+\mathbf{a} \in \mathbb{X}, \\ v(\mathbf{x}) + r & \text{else.} \end{cases}$$

  We denote by $\mathbb{T}^f$ the set of operators $\{T^t_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{A}}$.

- the choice operator

$$T^c_{\mathbf{a}} v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}), v(\mathbf{x}+\mathbf{a}) + c\} & \text{if } \mathbf{x}+\mathbf{a} \in \mathbb{X}. \\ v(\mathbf{x}) + r & \text{else.} \end{cases}$$

  We denote by $\mathbb{T}^c$ the set of operators $\{T^c_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{A}}$.

The operator A_i can be one of these two types, so $A_i \in \mathbb{T}^c \cup \mathbb{T}^f$. Note that the transition parameters p_i, the cost parameters of the events c and r, and the parameters of cost function C are components of $\mathbf{x}$.

With $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{A}$ the translation of one unit in direction i (the "1" is in the i^{th} position), Table 6.2 gives the link between our operators and some of the main operators of the literature (Koole, 1998, 2006; Çil et al., 2009).

Example 1: For the tandem queue problem, the MDP formulation is

$$\begin{cases} \mathcal{T}v & = & C + \mu_1 A_1 v + \mu_2 A_2 v + \lambda A_3 v + p_0 v, \quad \text{with} \\ C(\mathbf{x}) & = & x_1 h_1 + \max\{x_2, 0\} h_2 + \max\{-x_2, 0\} b, \\ A_1 v(\mathbf{x}) & = & \min(v(\mathbf{x}), v(\mathbf{x}+\mathbf{e}_1)) = T_{A(1)} v(\mathbf{x}) = T^c_{\mathbf{e}_1} v(\mathbf{x}), \\ A_2 v(\mathbf{x}) & = & \begin{cases} \min(v(\mathbf{x}), v(\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2)) & \text{if } x_1 > 0, \\ v(\mathbf{x}) & \text{else,} \end{cases}, \quad \text{and} \\ A_3 v(\mathbf{x}) & = & v(\mathbf{x}-\mathbf{e}_2) = T_{D(2)} v(\mathbf{x}) = T^t_{-\mathbf{e}_2} v(\mathbf{x}). \end{cases}$$

With $C(\mathbf{x})$ the cost function, A_1 the event related to the and of the production in Server 1, A_2 the event related to the and of the production in Server 2 and A_3 the event of demand at Buffer 2.

Example 2: In the same way, the MDP formulation of the admission control problem can be written

$$\begin{cases} \mathcal{T}v & = & C + \mu A_0 v + \sum_{i=1}^{3} \lambda_i A_i v + p_0 v, \quad \text{with} \\ C(\mathbf{x}) & = & xh, \\ A_0 v(\mathbf{x}) & = & v(\mathbf{x}-\mathbf{e}) = T_D v(\mathbf{x}) = T^t_{-\mathbf{e}} v(\mathbf{x}), \quad \text{and} \\ A_i v(\mathbf{x}) & = & \min(v(\mathbf{x}), v(\mathbf{x}+\mathbf{e}) - R_i) = T_A v(\mathbf{x}) = T^c_{\mathbf{e}} v(\mathbf{x}) \\ & & \forall i \in \{1,2,3\}, \quad \text{with} \quad R_1 \geq R_2 \geq R_3. \end{cases}$$

Name	Operator from the literature	With T^c and T^t
Arrival	$T_{A(i)}v(\mathbf{x}) = v(\mathbf{x}+\mathbf{e}_i)$	$T^t_{\mathbf{e}_i}v(\mathbf{x})$; $r=0$
Controlled arrival	$T_{CA(i)}v(\mathbf{x}) = \min\{v(\mathbf{x}); v(\mathbf{x}+\mathbf{e}_i)+c_a\}$	$T^c_{\mathbf{e}_i}v(\mathbf{x})$; $c=c_a$, $r=0$
Controlled arrival as fork	$T_{CAF}v(\mathbf{x}) = \min\{v(\mathbf{x}); v(\mathbf{x}+\sum_i \mathbf{e}_i)+c_a\}$	$T^c_{\sum_i \mathbf{e}_i}v(\mathbf{x})$; $c=c_a$, $r=0$
Routing	$T_{R(i,j)}v(\mathbf{x}) = \min_{k\in\{i,j\}} v(\mathbf{x}+\mathbf{e}_k)$	$T^c_{\mathbf{e}_j-\mathbf{e}_i}T^t_{\mathbf{e}_i}v(\mathbf{x})$; $c=r=0$
Batch arrival	$T_{BA(i)}v(\mathbf{x}) = \min_{0\geq j\geq B} v(\mathbf{x}+j\mathbf{e}_i+jc_a)$	$\left(\prod_{j=0}^{B} T^c_{\mathbf{e}_i}\right)v(\mathbf{x})$; $c=c_a$, $r=0$
Departure	$T_{D(i)}v(\mathbf{x}) = v((\mathbf{x}-\mathbf{e}_i)^+)$	$T^t_{-\mathbf{e}_i}v(\mathbf{x})$; $r=0$
Multi server departure	$T_{MD(i)}v(\mathbf{x}) = \gamma(\mathbf{x})v(\mathbf{x}-\mathbf{e}_i)+(1-\gamma(\mathbf{x}))v(\mathbf{x})$ with $\gamma(\mathbf{x}) = \frac{\min\{x_i,M\}}{M}$	$\gamma(\mathbf{x})T^t_{-\mathbf{e}_i}v(\mathbf{x})+(1-\gamma(\mathbf{x}))v(\mathbf{x})$; $r=0$
Controlled departure	$T_{CD(i)}v(\mathbf{x}) = \begin{cases}\min\{v(\mathbf{x}), v(\mathbf{x}-\mathbf{e}_i)+c_d\} \text{ if } x_i>0,\\ v(\mathbf{x}) \text{ else,}\end{cases}$	$T^c_{-\mathbf{e}_i}v(\mathbf{x})$; $c=c_d$, $r=0$
Parallel departure	$T_{PD}v(\mathbf{x}) = \sum_{i=1}^{l}\gamma_i v((\mathbf{x}-\mathbf{e}_i)^+)$ with $\sum_{i=1}^{l}\gamma_i = 1$	$\sum_{i=1}^{l}\gamma_i T^t_{-\mathbf{e}_i}v(\mathbf{x})$; $r=0$
Tandem server	$T_{T(i,j)}v(\mathbf{x}) = \begin{cases}v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j) \text{ if } x_i>0,\\ v(\mathbf{x}) \text{ else.}\end{cases}$	$T^t_{\mathbf{e}_j-\mathbf{e}_i}v(\mathbf{x})$; $r=0$
Controlled tandem server	$T_{CT(i,j)}v(\mathbf{x}) = \begin{cases}\min\{v(\mathbf{x}), v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j)+c_t\} \text{ if } x_i>0,\\ v(\mathbf{x}) \text{ else.}\end{cases}$	$T^c_{\mathbf{e}_j-\mathbf{e}_i}v(\mathbf{x})$; $c=c_t$, $r=0$

Table 6.2: Link with the literature.

Note that, the class of investigated MDPs can easily be extended to maximization problems (by replacing v by $-v$), to average cost criteria problems (average cost $g^\star$ is the limit of $\tau v^\star$ when $\tau \to 0$, see Weber and Stidham (1987)), or to finite horizon problems (with value function in k-th period: $v_k = \mathcal{T} v_{k-1}$).

6.4 Value function and state space properties

6.4.1 Value function properties

In this section we first present the main properties and notations to prove the structure of the optimal policy or a monotonic effect of a parameter on the optimal policy.

The operator $\Delta_{\boldsymbol{\alpha}}$ such that $\Delta_{\boldsymbol{\alpha}} v(\mathbf{x}) = v(\mathbf{x}+\boldsymbol{\alpha}) - v(\mathbf{x})$ and $\boldsymbol{\alpha} \in \mathbb{A} \cup \mathbb{E}$, is a discrete differentiation of v in the direction $\boldsymbol{\alpha}$. It could be seen as the expected marginal cost associated to the translation from $\mathbf{x}$ to $\mathbf{x}+\boldsymbol{\alpha}$. In the same way we define the operator Ω_A on v such as $\Omega_A v = (Av - v)$. Note that the quantity $\Delta_{\boldsymbol{\alpha}}\Omega_A$ is called the marginal benefit of the operator A in direction $\boldsymbol{\alpha}$.

In the following, the notation $v \geq 0$ means that for all $x \in \mathbb{X}$, $v(\mathbf{x}) \geq 0$, moreover we use the word increasing and decreasing to denote non-decreasing and non-increasing properties.

Definition 6.4.1. With $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (\mathbb{A} \cup \mathbb{E})^2$.

i) First order properties:

- v is $\mathtt{I}_{\boldsymbol{\alpha}}$ if $\Delta_{\boldsymbol{\alpha}} v \geq 0$ (increasing in the direction $\boldsymbol{\alpha}$),
- v is $\mathtt{D}_{\boldsymbol{\alpha}}$ if $\Delta_{\boldsymbol{\alpha}} v \leq 0$ (decreasing in the direction $\boldsymbol{\alpha}$),

ii) Second order properties:

- v is $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ if $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v \geq 0$ (supermodular in the directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$),
- v is $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ if $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v \leq 0$ (submodular in the directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$),
- v is $\mathtt{C}_{\boldsymbol{\alpha}}$ if $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha}} v \geq 0$ (convexity in the direction $\boldsymbol{\alpha}$),

iii) Marginal benefits properties:

- v is $\mathtt{IMB}(\boldsymbol{\alpha}, A)$ if $\Delta_{\boldsymbol{\alpha}}\Omega_A v \geq 0$ (increasing marginal benefit of A in the direction $\boldsymbol{\alpha}$),
- v is $\mathtt{DMB}(\boldsymbol{\alpha}, A)$ if $\Delta_{\boldsymbol{\alpha}}\Omega_A v \leq 0$ (decreasing marginal benefit of A in the direction $\boldsymbol{\alpha}$).

We say that an operator A propagates a property $\mathtt{P}$ with additional conditions if a sufficient condition to have Av with the property $\mathtt{P}$ is to have v with the property $\mathtt{P}$ and satisfied additional conditions. In this paper we prove properties on the optimal value function by induction. With $v_{k+1} = \mathcal{T} v_k$, Puterman (1994) proves that the only solution of the optimality equation $v^\star = \mathcal{T} v^\star$ is $v^\star = \lim_{k\to\infty} v_k$. So, if there exists a value function v_0 with the property $\mathtt{P}$ and if $\mathcal{T}$ propagates $\mathtt{P}$, then $v^\star$ has the property $\mathtt{P}$. Because the null value function $v_0 = 0$ has all the properties considered in this paper, in the following we will focus on the ability of $\mathcal{T}$ to propagate these properties.

We summarize some of the literature result on first and second order properties in Table 6.3 (Koole, 1998, 2006). This table use Boolean notations. With $\mathtt{P}$ a property on v, we write $\mathtt{P}$ as a Boolean value which is *true* (= 1) if and only if v has the property $\mathtt{P}$, and *false* (= 0) otherwise. In the same way, the Boolean value $|a|$ is *true* if and only if the assertion "a" is *true*. Note that we use the notation "." and "$\bigotimes$" (resp. "+" and "$\bigoplus$") to denote Boolean operator "and" (resp. "or"). Some properties of Boolean algebra are given in Appendix D.1.

Example 1: Veatch and Wein (1992) prove properties on the slopes of the optimal thresholds presented in figures 6.2 and 6.3. We reproduce here the sketch of their proof.

Let $\mathtt{P}$ be the property to be simultaneously $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$. The cost function C is $\mathtt{P}$ and the operators $T_{D(2)}$, $T_{A(1)}$, and $T_{CT(12)}$ propagate $\mathtt{P}$ (see Table 6.3). Because the operator $\mathcal{T}$ is a convex combination of functions with property $\mathtt{P}$, $\mathcal{T}$ propagates $\mathtt{P}$ and, by induction, the optimal value function $v^\star$ is $\mathtt{P}$. Then, the optimal policy is characterized by two switching curves $s_1^\star(x_2)$ and $s_2^\star(x_1)$ such that

- Produce at M_1 if and only if $x_1 < s_1^\star(x_2)$. Moreover $s_1^\star(x_2) - 1 \le s_1^\star(x_2+1) \le s_1^\star(x_2)$.
- Produce at M_2 if and only if $x_2 < s_2^\star(x_1)$. Moreover $s_2^\star(x_1) \le s_2^\star(x_1+1)$.

Note that a similar proof can be found in Section 3.4.

Example 2: In the same way we present the sketch of the proof of (Stidham, 1985).

Let $\mathtt{P}$ be the property to be both $\mathtt{C}_{\mathbf{e}}$ and $\mathtt{I}_{\mathbf{e}}$. The cost function C is $\mathtt{P}$ and the operators $T_{D(2)}$, $T_{A(1)}$, and $T_{CT(12)}$ propagate $\mathtt{P}$ (see Table 6.3). So, by induction, the optimal value function $v^\star$ is $\mathtt{P}$. Then, the optimal policy is characterized by thresholds t_i such that if $x \ge t_i$, it is optimal to reject an incoming class$-i$ customer; otherwise it is optimal to admit her. Moreover, if the cost are ordered as $R_1 \ge R_2 \ge R_3$, then $t_1 \ge t_2 \ge t_3$.

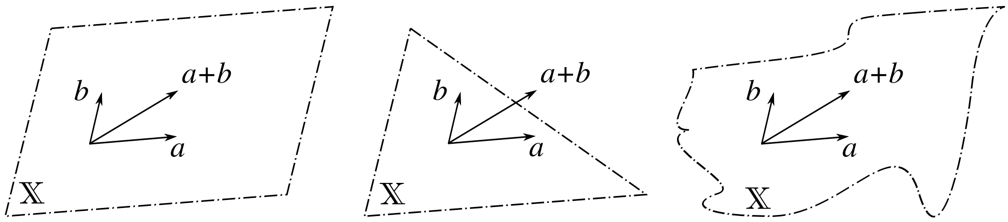
6.4.2 State space properties

Now, we present a property on the state space of the system. The figure 6.5 illustrates the following definition in three schematic examples of spaces $\mathbb{X}$.

Definition 6.4.2. With $(\mathbf{a}, \mathbf{b}) \in \mathbb{A}^2$, the state space $\mathbb{X}$ is $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ if and only if for all $\mathbf{x}$ such that $\{\mathbf{x}, \mathbf{x}+\mathbf{a}, \mathbf{x}+\mathbf{b}\}$ is a subset of $\mathbb{X}$, then $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is in $\mathbb{X}$.

Property	$T_{A(i)}$	$T_{CA(i)}$	$T_{D(i)}$	$T_{CD(i)}$	$T_{CT(i,j)}$
$\mathtt{I}_{\mathbf{e}_k}$	$true$	$true$	$true$	$\|i=k\|\,.\,\|0\leq c\|$ $+\|i\neq k\|$	$\|i=k\|\,.\,\|0\leq c\|\,.\mathtt{I}_{\mathbf{e}_j}$ $+\|i\neq k\|$
$\mathtt{S}_{\mathbf{e}_k,\mathbf{e}_l}$	$true$	$\|i=k\|$	$true$	$\|i\in\{k,l\}\|$	$\mathtt{S}_{\mathbf{e}_k-\mathbf{e}_l,\mathbf{e}_k}.\mathtt{S}_{\mathbf{e}_l-\mathbf{e}_k,\mathbf{e}_l}.\,\|i,j=k,l\|$
$\mathtt{S}_{-\mathbf{e}_k,\mathbf{e}_l}$	$true$	$\|i=k\|$	$true$	$\|i\in\{k,l\}\|$	0
$\mathtt{S}_{\mathbf{e}_k-\mathbf{e}_l,\mathbf{e}_k}$ $(l\neq k)$	$true$	$\mathtt{S}_{\mathbf{e}_k,\mathbf{e}_l}.\,\|i\in\{k,l\}\|$	$\|i=l\|\,.\mathtt{C}_{\mathbf{e}_k}+\|i=k\|\,.\mathtt{I}_{\mathbf{e}_k}$ $+\|i\notin\{k,l\}\|$	$\|i\in\{k,l\}\|\,.\mathtt{S}_{\mathbf{e}_k,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_l-\mathbf{e}_k,\mathbf{e}_l}.\,\|i,j=k,l\|$
$\mathtt{S}_{\mathbf{e}_k+\mathbf{e}_l,\mathbf{e}_k}$ $(l\neq k)$	$true$	$\mathtt{S}_{-\mathbf{e}_k,\mathbf{e}_l}.\,\|i\in\{k,l\}\|$	$\|i=l\|\,.\mathtt{C}_{\mathbf{e}_k}+\|i=k\|\,.\mathtt{I}_{\mathbf{e}_k}$ $+\|i\notin\{k,l\}\|$	$\|i\in\{k,l\}\|\,.\mathtt{S}_{\mathbf{e}_k,\mathbf{e}_i}$	$false$

Table 6.3: Additional condition to have [operator in column] which propagates [property in line], with Boolean notations ($true$ means that no additional conditions is needed and $false$ means that we can not conclude). From Koole (1998, 2006).



(a) $\mathbb{X}$ is $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ (b) $\mathbb{X}$ is not $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ (c) $\mathbb{X}$ is $\mathtt{R}_{\mathbf{a},\mathbf{b}}$

Figure 6.5: Examples of spaces $\mathbb{X}$ with property $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ (a) and (c), and without property $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ (b).

In some cases, like properties of the previous section, we use $\mathtt{R}_{\mathbf{a},\mathbf{b}}$ as a Boolean notation, with $\mathtt{R}_{\mathbf{a},\mathbf{b}} = 1$ if and only if $\mathbb{X}$ has property $\mathtt{R}_{\mathbf{a},\mathbf{b}}$.

Example 1: The state space of the tandem queue problem is $\mathbb{X} = (\mathbb{N} \times \mathbb{Z} \times \dots)$. So the state space is $\mathtt{R}_{\mathbf{e}_1,-\mathbf{e}_2}$ because for all $\{(x_1, x_2), (x_1 + 1, x_2), (x_1, x_2 - 1)\}$ subset of $\mathbb{X}$, (x_1+1, x_2-1) is in $\mathbb{X}$. However, the state space is not $\mathtt{R}_{-\mathbf{e}_1,-\mathbf{e}_1}$ because with $x_1 = 1$, $(x_1 - 1, x_2)$ is in the state space but $(x_1 - 2, x_2)$ is not.

Example 2: In the same way, the state space of the admission control problem $\mathbb{X} = (\mathbb{N} \times \dots)$ is $\mathtt{R}_{\mathbf{e},\mathbf{e}}$, but is not $\mathtt{R}_{-\mathbf{e},-\mathbf{e}}$.

6.5 Qualitative sensitivity analysis

With $\boldsymbol{\epsilon} \in \mathbb{E}$ a perturbation in the system value parameter such that $\mathbf{x}+\boldsymbol{\epsilon} \in \mathbb{X}$, in Example 2 we are interested in the evolution of the optimal thresholds t_i defined by

$$t_i = \min\{x | \Delta_{\mathbf{e}} v^\star(\mathbf{x}) - R_i \geq 0\}$$

when the system is perturbed:

$$t_i' = \min\{x | \Delta_{\mathbf{e}} v^\star(\mathbf{x} + \epsilon) - R_i - \epsilon_{R_i} \geq 0\}.$$

In this case, if $\Delta_{\mathbf{e}} v^\star(\mathbf{x}+\boldsymbol{\epsilon}) - R_i - \Delta_{\mathbf{e}} v^\star(\mathbf{x}) + R_i - \epsilon_{R_i} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{e}} v^\star(\mathbf{x}) - \epsilon_{R_i}$ is positive (resp. negative) for all $\mathbf{x}$, then t_i' is lower (resp. greater) than t_i.

In a general case, if the MDP formulation contains $\min\{v(\mathbf{x}), v(\mathbf{x} + \mathbf{d}) + c\}$, the aim is to find the condition for $\Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) + \epsilon_c$ to be positive for all $\mathbf{x}$. If ϵ_c is positive, it is equivalent to knowing the condition to have v supermodular in directions $\mathbf{d}$ and $\boldsymbol{\epsilon}$. Note that, when ϵ_c is negative, we can not conclude on the evolution of the optimal policy.

We do not consider condition to have $\Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) + \epsilon_c$ negative because it is equivalent to finding the condition to have $\Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) + \epsilon_c$ positive by replacing $\boldsymbol{\epsilon}$ by $-\boldsymbol{\epsilon}$ ($\mathtt{S}^{ub}_{\mathbf{d},\boldsymbol{\epsilon}} = \mathtt{S}_{\mathbf{d},-\boldsymbol{\epsilon}}$, see Appendix D.2).

In the rest of the paper we focus on the ability of $\mathcal{T}$ to propagates the property $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$

with $\mathbf{d} \in \mathbb{A}$ and $\boldsymbol{\epsilon} \in \mathbb{E}$. So we suppose that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v$ is positive and our aim is to find the condition for $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\mathcal{T}v$ to be positive.

$$\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x}) = \begin{pmatrix} \Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C(\mathbf{x}) \\ +\sum_{i=1}^{l}\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}p_iA_iv(\mathbf{x}) \\ +\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}p_0v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C(\mathbf{x}) \\ +\sum_{i=1}^{l}p_i\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}A_iv(\mathbf{x}) \\ +p_0\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{x}) \\ +\sum_{i=1}^{l}\epsilon_{p_i}\Delta_{\mathbf{d}}[A_iv(\mathbf{x})-v(\mathbf{x})] \end{pmatrix}$$

If $\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C$ is positive, A_i propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$, and $\epsilon_{p_i}\Delta_{\mathbf{d}}\Omega_{A_i}v$ is positive, then $\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v$ is positive. Knowing that the condition to be $\mathtt{IMB}(\mathbf{d}, A_i)$ is the same than the condition to be $\mathtt{DMB}(-\mathbf{d}, A_i)$ (see Appendix D.2) we can write our first theorem.

Theorem 6.5.1. *$\mathcal{T}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if*

$$|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C \geq 0| \,.\, \bigotimes_{i=1}^{l} \left[\begin{array}{c} |A_i \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}| \\ .\begin{pmatrix} |\epsilon_{p_i} < 0| \,.\mathtt{IMB}(-\mathbf{d}, A_i) \\ +|\epsilon_{p_i} > 0| \,.\mathtt{IMB}(\mathbf{d}, A_i) \\ +|\epsilon_{p_i} = 0| \end{pmatrix} \end{array} \right]$$

The condition $\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C \geq 0$ has to be tested for each cost function C. Our aim is to find a condition for A_i to propagate $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ and $\Delta_{\mathbf{d}}\Omega_{A_i}v \geq 0$. The following theorem considers this problem if $A_i \in \mathbb{T}^c \cup \mathbb{T}^f$.

Theorem 6.5.2.

- *$T^t_{\mathbf{a}}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if*

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d}}).(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_r \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d}}).$$

- *v is $\mathtt{IMB}(\mathbf{d}, T^t_{\mathbf{a}})$ if*

$$\mathtt{S}_{\mathbf{d},\mathbf{a}}.(|\Delta_{\mathbf{a}}v \leq r| \,.\mathtt{R}_{\mathbf{a},-\mathbf{d}} + |\Delta_{\mathbf{a}}v \geq r| \,.\mathtt{R}_{\mathbf{a},\mathbf{d}} + \mathtt{R}_{\mathbf{a},\mathbf{d}}.\mathtt{R}_{\mathbf{a},-\mathbf{d}}).$$

- *$T^c_{\mathbf{a}}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if*

$$\begin{bmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},\mathbf{a}}.\,|\epsilon_c \leq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \\ +\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},-\mathbf{a}}.\,|\epsilon_c \geq 0| + \mathtt{S}_{-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \end{bmatrix}$$
$$.(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| \,.\, |\epsilon_r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d}})$$
$$.(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0| \,.\, |\epsilon_r \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d}}).$$

- *v is $\mathtt{IMB}(\mathbf{d}, T^c_{\mathbf{a}})$ if*

$$\mathtt{S}_{\mathbf{d},\mathbf{a}}.(|r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d}}).\mathtt{R}_{\mathbf{a},-\mathbf{d}}.$$

The proof of Theorem 6.5.2 is given in D.4 for translation operator and D.5 for choice operator. Tables 6.4 and 6.5 give simplified applications of this theorem for the specific case $\epsilon_c = \epsilon_r = 0$.

Example 1:

Property 6.5.3. *For the tandem queue problem, the influence of the parameter variations on the optimal switching curves $s_1(x_2)$ and $s_2(x_1)$ is described in following table:*

Parameter (p)	*Increases ($\epsilon_p > 0$)*	*Decreases ($\epsilon_p < 0$)*
μ_1	*Can not conclude*	*Can not conclude*
μ_2	*Can not conclude*	*Can not conclude*
λ	*Increase*	*Decrease*
h_1	*Decrease if* $\epsilon_{h_1} \leq \min\{\epsilon_{h_2}, -\epsilon_b\}$	*Increase if* $\epsilon_{h_1} \geq \max\{\epsilon_{h_2}, -\epsilon_b\}$
h_2	*Decrease*	*Increase*
b	*Increase*	*Decrease*

To illustrate Proposition 6.5.3, Figure 6.6 presents a variation of the cost parameters. Because $0 \leq \epsilon_{h_1} \leq \min\{\epsilon_{h_2}, -\epsilon_b\}$, the optimal policy shows a monotonic behavior.

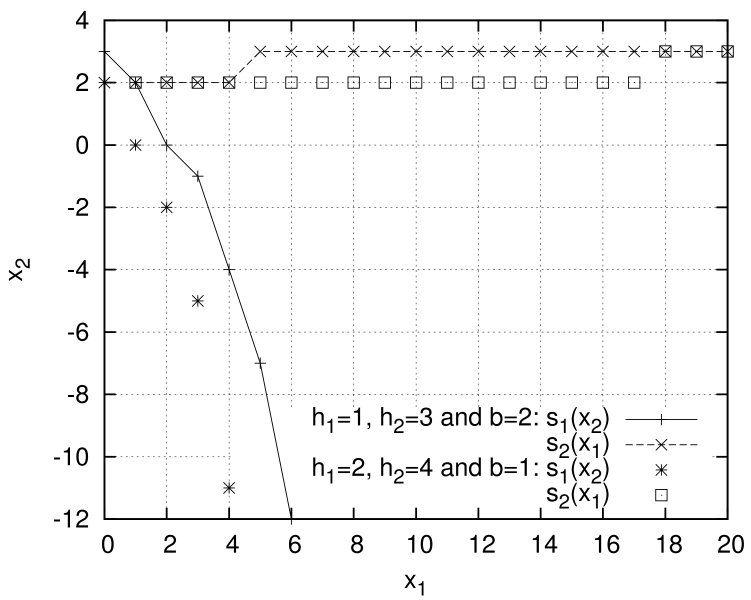


Figure 6.6: Effect of cost parameters on the optimal policy, ($\mu_1 = 2$, $\mu_2 = 1.3$, $\lambda = 1$).

Proof. Let $\epsilon = (0, 0, \epsilon_{\mu_1}, \epsilon_{\mu_2}, \epsilon_\lambda, \epsilon_{h_1}, \epsilon_{h_2}, \epsilon_b)$ be a perturbation on the system. According to

Property	$T_{A(i)}$	$T_{CA(i)}$, $T_{CD(i)}$	$T_{CT(i,j)}$	$T_{D(i)}$	$T_{T(i)}$
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	$true$	$true$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i-\mathbf{e}_j}.\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$true$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$true$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}.\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$true$	$\mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}}.(\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j}+\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}})$ $+\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}+\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}})$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}.\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	$true$	$true$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	$true$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}.\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}.\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$ $+\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}.\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$

Table 6.4: Simplified condition for [operator in column] to propagate [property in line] with $\epsilon_c = \epsilon_r = 0$ ($true$ means no additional condition needed).

Direction	$T_{A(i)}$, $T_{CA(i)}$	$T_{D(i)}$	$T_{CD(i)}$	$T_{T(i)}$	$T_{CT(i,j)}$
$\mathbf{e}_i$	$\mathtt{C}_{\mathbf{e}_i}$	$false$	$false$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}.\mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$	$false$
$\mathbf{e}_j-\mathbf{e}_i$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i}.\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{C}_{\mathbf{e}_j-\mathbf{e}_i}.\mathtt{I}_{\mathbf{e}_i-\mathbf{e}_j}$	$\mathtt{C}_{\mathbf{e}_j-\mathbf{e}_i}$
$\mathbf{e}_j$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\mathbf{e}_i+\mathbf{e}_j$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i}.\mathtt{I}_{-\mathbf{e}_i}$	$false$	$false$	$false$
$-\mathbf{e}_i$	$false$	$\mathtt{C}_{\mathbf{e}_i}.\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{C}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i-\mathbf{e}_j}.\mathtt{I}_{\mathbf{e}_i-\mathbf{e}_j}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i-\mathbf{e}_j}$
$\mathbf{e}_i-\mathbf{e}_j$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}.\mathtt{I}_{-\mathbf{e}_i}$	$false$	$false$	$false$
$-\mathbf{e}_j$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i-\mathbf{e}_j}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i-\mathbf{e}_j}$
$-\mathbf{e}_i-\mathbf{e}_j$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}.\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$false$	$false$

Table 6.5: Simplified condition for $\mathtt{IMB}$([direction in line], [operator in column]) $= true$ with $\epsilon_c = \epsilon_r = 0$ ($false$ means that we can not conclude).

the Theorem 6.5.1 the condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ is

$$\begin{aligned}|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C \geq 0| \,.\, \left|T^c_{\mathbf{e}_1} \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}\right| \,.\, \left|T^c_{\mathbf{e}_2-\mathbf{e}_1} \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}\right| \,.\, \left|T^t_{-\mathbf{e}_2} \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}\right| \\ .\left(|\epsilon_{\mu_1} < 0| \,.\mathtt{IMB}(-\mathbf{d}, T^c_{\mathbf{e}_1}) + |\epsilon_{\mu_1} > 0| \,.\mathtt{IMB}(\mathbf{d}, T^c_{\mathbf{e}_1}) + |\epsilon_{\mu_1} = 0|\right) \\ .\left(|\epsilon_{\mu_2} < 0| \,.\mathtt{IMB}(-\mathbf{d}, T^c_{\mathbf{e}_2-\mathbf{e}_1}) + |\epsilon_{\mu_2} > 0| \,.\mathtt{IMB}(\mathbf{d}, T^c_{\mathbf{e}_2-\mathbf{e}_1}) + |\epsilon_{\mu_2} = 0|\right) \\ .\left(|\epsilon_{\lambda} < 0| \,.\mathtt{IMB}(-\mathbf{d}, T^t_{-\mathbf{e}_2}) + |\epsilon_{\lambda} > 0| \,.\mathtt{IMB}(\mathbf{d}, T^t_{-\mathbf{e}_2}) + |\epsilon_{\lambda} = 0|\right).\end{aligned}$$

First we want to find condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$. With Theorem 6.5.2 the previous condition becomes

$$\begin{aligned}|\epsilon_{h_1} \geq 0| \,.true.true.\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}.\mathtt{S}_{\mathbf{e}_2,\boldsymbol{\epsilon}} \\ .\,(+\,|\epsilon_{\mu_1} > 0| \,.\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1} + |\epsilon_{\mu_1} = 0|) \\ .\,(|\epsilon_{\mu_2} < 0| \,.\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2} + |\epsilon_{\mu_2} > 0| \,.false + |\epsilon_{\mu_2} = 0|) \\ .\,(|\epsilon_{\lambda} < 0| \,.\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_1} + |\epsilon_{\lambda} > 0| \,.\mathtt{S}_{-\mathbf{e}_2,\mathbf{e}_1} + |\epsilon_{\lambda} = 0|)\,.\end{aligned}$$

Knowing that $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$ are *true* the condition reduces to

$$|\epsilon_{h_1} \geq 0| \,.\mathtt{S}_{\mathbf{e}_2,\boldsymbol{\epsilon}}.\, |\epsilon_{\mu_1} \geq 0| \,.\, |\epsilon_{\mu_2} \leq 0| \,.\, |\epsilon_{\lambda} \leq 0| \;\;. \tag{6.1}$$

In equation (6.1), we need that $\mathcal{T}$ propagates the property $\mathtt{S}_{\mathbf{e}_2,\boldsymbol{\epsilon}}$. As previously, the simplified condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{e}_2,\boldsymbol{\epsilon}}$ is

$$|\min\{\epsilon_{h_2}, -\epsilon_b\} \geq 0| \,.\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}.\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}.\, |\epsilon_{\mu_1} \geq 0| \,.\, |\epsilon_{\mu_2} \geq 0| \,.\, |\epsilon_{\lambda} \leq 0|\,. \tag{6.2}$$

Again, in (6.2) we need that $\mathcal{T}$ propagates $\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$. As previously the simplified condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$ is

$$|\epsilon_{h_1} \leq \min\{\epsilon_{h_2}, -\epsilon_b\}| \,.\mathtt{S}_{\mathbf{e}_2,\boldsymbol{\epsilon}}.\, |\epsilon_{\mu_1} \leq 0| \,.\, |\epsilon_{\mu_2} \geq 0| \,.\, |\epsilon_{\lambda} \leq 0|\,. \tag{6.3}$$

Conclusion, by aggregation of (6.1), (6.2) and (6.3) we obtain,

$$|0 \leq \epsilon_{h_1} \leq \min\{\epsilon_{h_2}, -\epsilon_b\}| \,.\, |\epsilon_{\mu_1} = 0| \,.\, |\epsilon_{\mu_2} = 0| \,.\, |\epsilon_{\lambda} \leq 0|\,.$$

So we can justified the behavior observed in Figure 6.2 because the two optimal switching curves $s_1(x_2)$ and $s_2(x_1)$ are decreasing if the rate of demand λ increases. Note that for condition of submodularity if suffice to replace $\boldsymbol{\epsilon}$ by $-\boldsymbol{\epsilon}$ to conclude that $s_1(x_2)$ and $s_2(x_1)$ are increasing if the rate of demand λ decreases. □

Example 2:

Property 6.5.4. *For the admission control problem, the influence of the parameter variations on the optimal thresholds t_i is described in following table:*

Parameter (p)	*Increases* ($\epsilon_p > 0$)	*Decreases* ($\epsilon_p < 0$)
λ_i	*Decrease*	*Increase*
μ	*Increase*	*Decrease*
h	*Decrease*	*Increase*
R_i	*Can not conclude*	*Can not conclude*

Table 6.6 illustrates Proposition 6.5.4. In this table, Instances 5, 6, 7, and 8 presents the variation of respectively μ, λ_1, h, and R_1. We can observe that Instance 5, 6, and 7 have monotonic behavior when compared to Instance I because the perturbations are consistent with Proposition 6.5.4. Moreover, Instance 8 has non-monotonic behavior because t_1 decreases and t_2 increases. We present this instance to stress that v can be $\mathtt{S}_{\epsilon,\mathbf{e}}$ (when $\epsilon_{R_i} \geq 0$) while the optimal policy has non-monotonic behavior. Note that the behaviors of Instances 5 and 6, (variation of μ and λ_i) have already been proven by Çil et al. (2009).

Instance	μ	λ_1	λ_2	λ_3	h	R_1	R_2	R_3	t_1	t_2	t_3
1	1	0.6	0.6	0.6	1	-30	-20	-10	9	3	1
5	0.6	0.6	0.6	0.6	1	-30	-20	-10	4	1	0
6	1	1	0.6	0.6	1	-30	-20	-10	6	2	0
7	1	0.6	0.6	0.6	2	-30	-20	-10	5	2	1
8	1	0.6	0.6	0.6	1	-25	-20	-10	7	4	1

Table 6.6: Optimal thresholds in function of arrival rates.

The numerical results presented in introduction (see Table 6.1) are still not explained. It is the purpose of the next section.

Proof. Let $\epsilon = (0, \epsilon_{\lambda_1}, \epsilon_{\lambda_3}, \epsilon_{\lambda_3}, \epsilon_\mu, \epsilon_h, \epsilon_{R_1}, \epsilon_{R_2}, \epsilon_{R_3})$ be a perturbation on the system. According to the Theorem 6.5.1 the condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{e},\epsilon}$ is

$$\begin{gathered}|\Delta_{\mathbf{e}}\Delta_{\epsilon}C \geq 0| \,.\, \left|T^t_{-\mathbf{e}} \text{ propagates } \mathtt{S}_{\mathbf{e},\epsilon}\right| \,.\, \textstyle\bigotimes_{i=1}^{3} |T^c_{\mathbf{e}} \text{ propagates } \mathtt{S}_{\mathbf{e},\epsilon}| \\ .\, \left(|\epsilon_\mu < 0| \,.\mathtt{IMB}(-\mathbf{e}, T^t_{-\mathbf{e}}) + |\epsilon_\mu > 0| \,.\mathtt{IMB}(\mathbf{e}, T^t_{-\mathbf{e}}) + |\epsilon_\mu = 0|\right) \\ .\, \textstyle\bigotimes_{i=1}^{3} \left(|\epsilon_{\lambda_i} < 0| \,.\mathtt{IMB}(-\mathbf{e}, T^c_{\mathbf{e}}) + |\epsilon_{\lambda_i} > 0| \,.\mathtt{IMB}(\mathbf{e}, T^c_{\mathbf{e}}) + |\epsilon_{\lambda_i} = 0|\right).\end{gathered}$$

Using Theorem 6.5.2, the previous equation becomes

$$\begin{gathered}|\epsilon_h \geq 0| \,.true.\, \textstyle\bigotimes_{i=1}^{3}(\mathtt{S}_{\mathbf{0},\epsilon}.\mathtt{C}_{\mathbf{e}}.\, |\epsilon_{R_i} \geq 0| \, \mathtt{S}_{\mathbf{e},\epsilon}.\, |\epsilon_{R_i} = 0| + false) \\ .\, (|\epsilon_\mu < 0| \,.\mathtt{C}_{\mathbf{e}}.\mathtt{I}_{\mathbf{e}} + |\epsilon_\mu > 0| \,.false + |\epsilon_\mu = 0|) \\ .\, \textstyle\bigotimes_{i=1}^{3} \left(|\epsilon_{\lambda_i} < 0| \,.false + |\epsilon_{\lambda_i} > 0| \,.\mathtt{C}_{\mathbf{e}} + |\epsilon_{\lambda_i} = 0|\right).\end{gathered}$$

Knowing that v is $\mathtt{C}_{\mathbf{e}}$ and $\mathtt{I}_{\mathbf{e}}$, the previous equation reduces to

$$|\epsilon_h \geq 0| \,.\, |\epsilon_\mu \leq 0| \,.\, \textstyle\bigotimes_{i=1}^{3} \left(|\epsilon_{\lambda_i} \geq 0| \,.\, |\epsilon_{R_i} \geq 0|\right) \tag{6.4}$$

If the previous condition is *true*, v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}}$. To conclude on the condition on ϵ to have the thresholds t_i decreasing we need to have $\Delta_{\mathbf{e}}v(\mathbf{x}+\boldsymbol{\epsilon}) - \Delta_{\mathbf{e}}v(\mathbf{x}) - \epsilon_{R_i}$ positive. So we need v with property $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ and ϵ_{R_i} positive. Finally the equation 6.4 reduces to

$$|\epsilon_h \geq 0| \,.\, |\epsilon_\mu \leq 0| \,.\, \bigotimes_{i=1}^{3} \left(|\epsilon_{\lambda_i} \geq 0| \,.\, |\epsilon_{R_i} = 0|\right) \tag{6.5}$$

Note that the conditions to have the thresholds t_i increasing can be obtain by replacing $\boldsymbol{\epsilon}$ by $-\boldsymbol{\epsilon}$ in the equation (6.5). □

6.6 Compensation

In Example 2, according to Proposition 6.5.4, we know that ϵ_{λ_1} positive tends to make v $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ and ϵ_{λ_2} negative tends to make v $\mathtt{S}_{\mathbf{d},-\boldsymbol{\epsilon}}$. So now, we can not conclude on the behavior of the optimal policy if ϵ_{λ_1} is positive and ϵ_{λ_2} negative at same time. In this section we want to extend the Theorem 6.5.1 to consider this situation and prove that v is still $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$. We call this phenomenon compensations between perturbations.

We do not find compensations if the events are different, so our results are limited to the sum of similar operator (either choice or translation) with same vector $\mathbf{a}$ and only one cost different. So, we define a set of operator indexes I such that for all indexes i and j in I, A_i is the same operator than A_j (e.g. $A_i = A_j = T^c_{\mathbf{a}}$) and the costs are equals except for one which is increasing with i (e.g. $r_i = r_j$ and $r_i \leq r_j$ if $i < j$). In Example 2, I can be equal to $\{1,2,3\}$, $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{1\}$, $\{2\}$, $\{3\}$, or $\{\emptyset\}$ because the MDP formulation contains $\sum_{i=1}^{3} A_i v$ with $A_i v(\mathbf{x}) = \min\{v(\mathbf{x}), v(\mathbf{x}+1) - R_i\}$ and $-R_1 \leq -R_2 \leq -R_3$.

With the translation operator and the choice operator, we have three types of set I:

i) $\forall (i,j) \in I^2$: $A_i = A_j = T^t_{\mathbf{a}}$ and $r_i \leq r_j$ if $i < j$,

ii) $\forall (i,j) \in I^2$, $A_i = A_j = T^c_{\mathbf{a}}$, $c_i = c_j$, and $r_i \leq r_j$ if $i < j$,

iii) $\forall (i,j) \in I^2$, $A_i = A_j = T^c_{\mathbf{a}}$, $r_i = r_j$, and $c_i \leq c_j$ if $i < j$,

For a given I, the following theorem presents the condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$. We can observe that when $I = \emptyset$, it reduces to Theorem 6.5.1.

Theorem 6.6.1. *With* $\mathcal{T}v = C + \sum_{i\in I} p_i A_i v + \sum_{i\notin I} p_i A_i v + p_0 v$, $\mathcal{T}$ *propagates* $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ *if*

$$
\begin{aligned}
&|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}} C \geq 0| \,.\, \bigotimes_{i=0}^{l} |A_i \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}| \\
&.\bigotimes_{i\notin I} \left(\mathtt{IMB}(\mathbf{d}, A_i).\, |\epsilon_{p_i} \geq 0| + \mathtt{IMB}(-\mathbf{d}, A_i).\, |\epsilon_{p_i} \leq 0| + |\epsilon_{p_i} = 0|\right) \\
&.\bigotimes_{i\in I} \left[\begin{array}{r} \mathtt{IMB}(\mathbf{d}, A_i).\left|\sum_{k\in I} \epsilon_{p_k} \geq 0\right| . \bigotimes_{j\in I, i<j} \begin{pmatrix} \left|\Delta_{\mathbf{d}}\Omega_{A_i} v \leq \Delta_{\mathbf{d}}\Omega_{A_j} v\right| . \left|\epsilon_{p_i} \leq \epsilon_{p_j}\right| \\ + \left|\Delta_{\mathbf{d}}\Omega_{A_i} v \geq \Delta_{\mathbf{d}}\Omega_{A_j} v\right| . \left|\epsilon_{p_j} \leq \epsilon_{p_i}\right| \end{pmatrix} \\ + \mathtt{IMB}(-\mathbf{d}, A_i).\left|\sum_{k\in I} \epsilon_{p_k} \leq 0\right| . \bigotimes_{j\in I, i<j} \begin{pmatrix} \left|\Delta_{-\mathbf{d}}\Omega_{A_i} v \leq \Delta_{-\mathbf{d}}\Omega_{A_j} v\right| . \left|\epsilon_{p_i} \geq \epsilon_{p_j}\right| \\ + \left|\Delta_{-\mathbf{d}}\Omega_{A_i} v \geq \Delta_{-\mathbf{d}}\Omega_{A_j} v\right| . \left|\epsilon_{p_j} \geq \epsilon_{p_i}\right| \end{pmatrix} \end{array} \right]
\end{aligned}
$$

Proof. The relation $\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}} \left[\sum_{i\in I} p_i A_i v + p_0 v\right] \geq 0$ is *true* if $\forall i \in I$ A_i propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ and if $\sum_{i\in I} \epsilon_{p_i} \Delta_{\mathbf{d}}\Omega_{A_i} v \geq 0$. The sufficient condition to have $\sum_{i\in I} \epsilon_{p_i} \Delta_{\mathbf{d}}\Omega_{A_i} v \geq 0$ is $0 \leq \Delta_{\mathbf{d}}\Omega_{A_i} v \leq \Delta_{\mathbf{d}}\Omega_{A_j} v$ with $\forall i < j$, $\sum_{i\in I} \epsilon_{p_i} \geq 0$, and $\epsilon_{p_i} \leq \epsilon_{p_j}$ (see Appendix D.3). □

Let $\boldsymbol{\epsilon}_r \in \mathbb{E}$ (resp. $\boldsymbol{\epsilon}_c \in \mathbb{E}$) be perturbation applied only on the cost r (resp. c). In Theorem 6.6.1, we need to know the conditions to have $\Delta_{\mathbf{d}}\Omega_{A_i} v \leq \Delta_{\mathbf{d}}\Omega_{A_j} v$ with i and j in I. In our case, this problem is equivalent to know the condition to have $\Delta_{\boldsymbol{\epsilon}_c}\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a}}} v$ positive , $\Delta_{\boldsymbol{\epsilon}_r}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a}}} v$ positive , and $\Delta_{\boldsymbol{\epsilon}_c}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a}}} v$ positive. This is the purpose of the next theorem.

Theorem 6.6.2.

- $\Delta_{\boldsymbol{\epsilon}_c}\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a}}} v \geq 0$ *if:* $|\epsilon_c \geq 0| \,.\mathtt{R}_{\mathbf{a},-\mathbf{d}} + |\epsilon_c \leq 0| \,.\mathtt{R}_{\mathbf{a},\mathbf{d}}$,
- $\Delta_{\boldsymbol{\epsilon}_r}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a}}} v \geq 0$ *if:* $|\epsilon_c \geq 0| \,.\mathtt{R}_{\mathbf{a},-\mathbf{d}} + |\epsilon_c \leq 0| \,.\mathtt{R}_{\mathbf{a},\mathbf{d}}$,
- $\Delta_{\boldsymbol{\epsilon}_c}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a}}} v \geq 0$ *if:* $|\epsilon_c \geq 0| \,.\mathtt{S}_{-\mathbf{d},\mathbf{a}}.\mathtt{R}_{\mathbf{a},\mathbf{d}} + |\epsilon_c \leq 0| \,.\mathtt{S}_{\mathbf{d},\mathbf{a}}.\mathtt{R}_{\mathbf{a},-\mathbf{d}}$.

The proof of this theorem is given in Appendix D.4 for translation operator and Appendix D.5 for choice operator.

Example 2: The following proposition extends the results of Section 6.5 about Example 2.

Property 6.6.3. *The thresholds t_i decrease if*

$$
\begin{array}{ll}
|\epsilon_h \geq 0| \,.\, |\epsilon_{R_1} = \epsilon_{R_2} = \epsilon_{R_3} = 0| \,.\, |\epsilon_\mu \leq 0| & \\
.\left[\begin{array}{c}
|\epsilon_{\lambda_1} + \epsilon_{\lambda_2} + \epsilon_{\lambda_3} \geq 0| \,.\, |\epsilon_{\lambda_1} \geq \epsilon_{\lambda_2} \geq \epsilon_{\lambda_3}| \\
+ |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \geq 0| \,.\, |\epsilon_{\lambda_1} \geq \epsilon_{\lambda_2}| \,.\, |\epsilon_{\lambda_3} \geq 0| \\
+ |\epsilon_{\lambda_2} + \epsilon_{\lambda_3} \geq 0| \,.\, |\epsilon_{\lambda_2} \geq \epsilon_{\lambda_3}| \,.\, |\epsilon_{\lambda_1} \geq 0| \\
+ |\epsilon_{\lambda_1} + \epsilon_{\lambda_3} \geq 0| \,.\, |\epsilon_{\lambda_1} \geq \epsilon_{\lambda_3}| \,.\, |\epsilon_{\lambda_2} \geq 0| \\
+ |\epsilon_{\lambda_1} \geq 0| \,.\, |\epsilon_{\lambda_2} \geq 0| \,.\, |\epsilon_{\lambda_3} \geq 0|
\end{array}\right]
&
\begin{array}{c}
(\textit{if } I = \{1,2,3\}) \\
(\textit{if } I = \{1,2\}) \\
(\textit{if } I = \{2,3\}) \\
(\textit{if } I = \{1,3\}) \\
(\textit{if } I = \emptyset, \{1\}, \{2\}, \textit{ or } \{3\})
\end{array}
\end{array}
$$

Proposition 6.6.3 justifies the behavior of Instances 2 and 3 in the introduction of this chapter (see Table 6.1). Note that we can not conclude about the behavior of Instance 4 because the perturbation does not respect the condition given in Theorem 6.6.3. In Table 6.7 we extends the numerical study of the Example 2 to illustrate Proposition 6.6.3.

I	λ_1	λ_2	λ_3	t_1	t_2	t_3
	0.6	0.6	0.6	9	3	1
$\{1,2,3\}$	1	0.7	0.1	6	2	0
$\{1,2\}$	0.8	0.4	0.7	7	3	1
$\{2,3\}$	0.7	0.8	0.4	8	3	0
$\{1,3\}$	0.8	1	0.4	7	2	0
$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$	1	0.6	0.6	6	2	0

Table 6.7: Optimal thresholds as a function of arrival rates ($\mu = h_1 = 1$, $R_1 = 30$, $R_2 = 20$, and $R_3 = 10$).

Proof. We present the case $I = \{1,2\}$, corresponding to the second line of the brackets. The rest of the proof is given is Appendix D.6. Note that the last line of the bracket has already been proven in Section 6.5. According to the Theorem 6.6.1 the condition for $\mathcal{T}$ to propagate $\mathtt{S}_{\mathbf{e},\boldsymbol{\epsilon}}$ is

$$
\begin{array}{c}
|\Delta_{\mathbf{e}}\Delta_{\boldsymbol{\epsilon}} C \geq 0| \,.\, \bigotimes_{i\in\{0,1,2,3\}} |A_i \text{ propagates } \mathtt{S}_{\mathbf{e},\boldsymbol{\epsilon}}| \\
.\,(|\epsilon_\mu < 0| \,.\mathtt{IMB}(-\mathbf{e}, A_0) + |\epsilon_\mu > 0| \,.\mathtt{IMB}(\mathbf{e}, A_0) + |\epsilon_\mu = 0|) \\
.\,(|\epsilon_{\lambda_3} < 0| \,.\mathtt{IMB}(-\mathbf{e}, A_3) + |\epsilon_{\lambda_3} > 0| \,.\mathtt{IMB}(\mathbf{e}, A_3) + |\epsilon_{\lambda_3} = 0|) \\
.\left(\begin{array}{c}
|0 \leq \Delta_{\mathbf{e}}\Omega_{A_1} v \leq \Delta_{\mathbf{e}}\Omega_{A_2} v| \,.\, |\epsilon_{\lambda_1} \leq \epsilon_{\lambda_2}| \,.\, |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \geq 0| \\
+ |0 \leq \Delta_{\mathbf{e}}\Omega_{A_2} v \leq \Delta_{\mathbf{e}}\Omega_{A_1} v| \,.\, |\epsilon_{\lambda_2} \leq \epsilon_{\lambda_1}| \,.\, |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \geq 0| \\
+ |0 \leq \Delta_{\mathbf{e}}\Omega_{A_1} v \leq \Delta_{\mathbf{e}}\Omega_{A_2} v| \,.\, |\epsilon_{\lambda_1} \geq \epsilon_{\lambda_2}| \,.\, |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \leq 0| \\
+ |0 \leq \Delta_{\mathbf{e}}\Omega_{A_2} v \leq \Delta_{\mathbf{e}}\Omega_{A_1} v| \,.\, |\epsilon_{\lambda_2} \geq \epsilon_{\lambda_1}| \,.\, |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \leq 0|
\end{array}\right)
\end{array}
$$

Knowing that the sufficient condition to have $|0 \leq \Delta_{\mathbf{e}}\Omega_{A_2} v \leq \Delta_{\mathbf{e}}\Omega_{A_1} v|$ is *true*, and knowing that v is $\mathtt{C}_{\mathbf{e}}$ and $\mathtt{I}_{\mathbf{e}}$ (see Example 2 in Section 6.5), the condition to have the thresholds

t_i decreasing reduces to

$$|\epsilon_h \geq 0| \,.\, |\epsilon_{R_1} = \epsilon_{R_2} = \epsilon_{R_3} = 0| \,.\, |\epsilon_\mu \leq 0| \,.\, |\epsilon_{\lambda_1} \leq \epsilon_{\lambda_2}| \,.\, |\epsilon_{\lambda_1} + \epsilon_{\lambda_2} \geq 0| \,.\, |\epsilon_{\lambda_3} \geq 0|$$

□

6.7 Extensions

6.7.1 Nested operators

A combination of operators like $[T^c_{\mathbf{a}}(T^t_{\mathbf{b}}v)](\mathbf{x})$ is called nested operator. It will be written as a non commutative product of operators on v: $T^c_{\mathbf{a}}T^t_{\mathbf{b}}v$. We denote the set of nested operators

$$\mathbb{T} = \{A_1A_2\dots A_l\}_{l\in\mathbb{N},A_i\in\mathbb{T}^c\cup\mathbb{T}^f} = \left\{\prod_{i=1}^{l} A_i\right\}_{l\in\mathbb{N},A_i\in\mathbb{T}^c\cup\mathbb{T}^f}.$$

Some operators of the literature can be written as a nesting of operators like the operator of routing

$$T_{R(i,j)}v(\mathbf{x}) = \min_{k\in\{i,j\}} v(\mathbf{x}+\mathbf{e}_k) = T^c_{\mathbf{e}_j-\mathbf{e}_i}T^t_{\mathbf{e}_i}v(\mathbf{x}) \quad \text{with} \quad c = r = 0,$$

and the operator of arrival by batch

$$T_{BA(i)}v(\mathbf{x}) = \min_{0\geq j\geq B} v(\mathbf{x}+j\mathbf{e}_i+jp) = \left(\prod_{j=0}^{B} T^c_{\mathbf{e}_i}\right) v(\mathbf{x}) \quad \text{with} \quad c = p, r = 0.$$

Theorem 6.7.1. *If $\forall i \in \{1, l\}$, A_i propagates* P *with the additional condition $\mathcal{C}_i$, then $\prod_{i=1}^{l} A_i$ propagates* P *if all $\mathcal{C}_i$ are true (i.e. $\bigotimes_{i=1}^{l} |\mathcal{C}_i| = 1$).*

Proof. Suppose v has the property P.

1) A_1v propagates P if $|\mathcal{C}_1| = 1$

2) A_2A_1v propagates P if $|\mathcal{C}_2| \,.\, |\mathcal{C}_1| = 1$

-) ...

l) $\prod_{i=1}^{l} A_i$ propagates P if $\bigotimes_{i=1}^{l} |\mathcal{C}_i|$.

□

Example 1 For tandem queue problem, if the productions are made by batches, the MDP formulation becomes

$$\begin{cases} \mathcal{T}v = C + \mu_1 \prod_{i=0}^{B_1} T^c_{\mathbf{e}_1}v + \mu_2 \prod_{i=0}^{B_2} T^c_{\mathbf{e}_2-\mathbf{e}_1}v + T^t_{-\mathbf{e}_2}v + p_0v, \\ C(\mathbf{x}) = x_1h_1 + \max\{x_2, 0\}h_2 + \max\{-x_2, 0\}b. \end{cases}$$

With this formulation, all previous results pertain, so v is $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$, and $\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$.

Proof. Let $\mathtt{P}$ be the property to be at the same time $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$, and $\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$. Because C is $\mathtt{P}$ and operators $T^c_{\mathbf{e}_1}$, $T^c_{\mathbf{e}_2-\mathbf{e}_1}$ and $T^t_{-\mathbf{e}_2}$ propagates $\mathtt{P}$; then $v^\star$ is $\mathtt{P}$. □

All the conditions are sufficient conditions, so the result of $\bigotimes_{i=1}^{l} |\mathcal{C}_i|$ could be very restrictive. For instance, with $\{x_i\} = \mathbb{N}$, the nested operator $T^t_{-\mathbf{e}_i} T^t_{\mathbf{e}_i}$ will have a very restrictive condition (conditions of $T^t_{-\mathbf{e}_i}$ and conditions of $T^t_{\mathbf{e}_i}$) but it is evident that there is no condition because it is equivalent to $T^t_{\mathbf{0}}$. In the following we want to find simplifications of condition for nested operators $T^c_{\mathbf{a}} T^t_{\mathbf{b}}$ and $T^t_{\mathbf{a}} T^t_{\mathbf{b}}$.

We define two special nested operators. With $\mathbf{a}, \mathbf{b} \in \mathbb{A}^2$ such that $\mathbb{X}$ is $\mathtt{R}(\mathbf{b})$ (i.e. $\forall \mathbf{x}$, $\mathbf{x} + \mathbf{b} \in \mathbb{X}$), let the nested translation operator be

$$T^t_{\mathbf{a},\mathbf{b}} = T^t_{\mathbf{a}} T^t_{\mathbf{b}} v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{a}+\mathbf{b}) & \text{if } \mathbf{x}+\mathbf{a}+\mathbf{b} \in \mathbb{X}, \\ v(\mathbf{x}+\mathbf{b}) + r & \text{else,} \end{cases}$$

and let the nested choice operator be

$$T^c_{\mathbf{a},\mathbf{b}} = T^c_{\mathbf{a}} T^t_{\mathbf{b}} v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}+\mathbf{b}), v(\mathbf{x}+\mathbf{a}+\mathbf{b}) + c\} & \text{if } \mathbf{x}+\mathbf{a}+\mathbf{b} \in \mathbb{X}, \\ v(\mathbf{x}+\mathbf{b}) + r & \text{else.} \end{cases}$$

These two operators generalized the choice operator and the translation operator with $b = 0$. Before giving the condition of propagation and increasing marginal benefits, we need to generalize the property $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2}$. With $(\mathbf{a}_1, \dots, \mathbf{a}_l, \mathbf{b}) \in \mathbb{A}^{l+1}$, the state space $\mathbb{X}$ is $\mathtt{R}_{\mathbf{a}_1,\dots,\mathbf{a}_l}(\mathbf{b})$ if and only if for all $\mathbf{x}$ such that $\{\mathbf{x}, \mathbf{x}+\mathbf{a}_1, \dots, \mathbf{x}+\mathbf{a}_l\} \subset \mathbb{X}$, then $\mathbf{x}+\mathbf{b} \in \mathbb{X}$. This property generalizes the property $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2}$ because $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2}$ is equivalent to $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2}(\mathbf{a}_1+\mathbf{a}_2)$ and $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2}$ implies $\mathtt{R}_{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_l}(\mathbf{a}_1+\mathbf{a}_2)$.

In the following theorem we generalize Theorems 6.5.2 and 6.6.2 to operators $T^c_{\mathbf{a},\mathbf{b}}$ and $T^t_{\mathbf{a},\mathbf{b}}$.

Theorem 6.7.2. • *Nested translation operator:*

– $T^t_{\mathbf{a},\mathbf{b}}$ *propagates* $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ *if*

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d})).(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_r \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})),$$

– v *is* $\texttt{IMB}(\mathbf{d}, T^t_{\mathbf{a},\mathbf{b}})$ *if*

$$\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}.\left(\begin{array}{c} \{\Delta_{\mathbf{a}} v \leq r\}.(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}).\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}) \\ +\{\Delta_{\mathbf{a}} v \geq r\}.(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}).\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}) \\ +\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}).\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}) \end{array}\right),$$

– $\Delta_{\epsilon_r}\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a},\mathbf{b}}} v \geq 0$ *if*

$$|\epsilon_r \geq 0|\,.\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}) + |\epsilon_r \leq 0|\,.\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}),$$

- *Nested choice operator:*

– $T^c_{\mathbf{a},\mathbf{b}}$ *propagates* $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ *if*

$$\begin{array}{c} \left[\begin{array}{c} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},\mathbf{a}}.\,|\epsilon_c \leq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \\ +\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},-\mathbf{a}}.\,|\epsilon_c \geq 0| + \mathtt{S}_{-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \end{array}\right] \\ .(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|\,.\,|\epsilon_r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d})) \\ .(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|\,.\,|\epsilon_r \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})), \end{array}$$

– $\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}} v \geq 0$ *if*

$$\mathtt{S}_{\mathbf{d},\mathbf{a}}.\mathtt{S}_{\mathbf{d},\mathbf{b}}.(|r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d})).\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}),$$

– $\Delta_{\epsilon_r}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}} v \geq 0$ *if*

$$|\epsilon_r \geq 0|\,.\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}) + |\epsilon_r \leq 0|\,.\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}),$$

– $\Delta_{\epsilon_c}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}} v \geq 0$ *if*

$$|\epsilon_c \geq 0|\,.\mathtt{S}_{\mathbf{d},-\mathbf{a}}.\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}) + |\epsilon_c \leq 0|\,.\mathtt{S}_{\mathbf{d},\mathbf{a}}.\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}).$$

The proof of this theorem is given in Appendix D.

6.7.2 Gamma operator

With a new definition of $\mathbf{x}$ as a generic vector in $\mathbb{X} \subset \mathbb{R}^n$, we define a new operator on v, called Gamma operator, which allows the variation of the service rate in function of $\mathbf{x}$,

$$\Gamma^A v(\mathbf{x}) = \gamma(\mathbf{x})Av(\mathbf{x}) + [1-\gamma(\mathbf{x})]v(\mathbf{x}).$$

The set of Gamma operator is noted $\mathbb{G}$. Moreover, we extend our problem to the generic formulation,

$$\mathcal{T}v(\mathbf{x}) = C(\mathbf{x}) + \sum_{i=1}^{l} \mu_i \Gamma_i^{A_i} v(\mathbf{x}) \text{ with } \{A_i, \Gamma_i\} \in \mathbb{T} \times \mathbb{G},\ \forall i \in \{1, \ldots, l\}.$$

With this generic formulation, sensitivity analysis can be done by adding a dimension to the vector $\mathbf{x}$ making functions $\gamma_i(\mathbf{x})$ varying in this dimension. Because it can consider any functions γ_i, this formulation allows even more. For instance, functions may vary depending on the status of queues, allowing to model multi-server, environment variables, and pricing.

We want to find the condition for $\mathcal{T}$ to propagate the property $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. By induction, we suppose that $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v \geq 0$ and we want to find the conditions to have $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\mathcal{T} v \geq 0$, for all Γ_i, A_i and v.

Theorem 6.7.3. *With Boolean notation a sufficient condition to have $\mathcal{T}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ is*

$$|\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} C \geq 0| \bigotimes_{i=1}^{l} \left[\begin{array}{l} |A_i \to \mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}| \,.\, |\gamma_i(\mathbf{x}) \in [0,1]| \\ \cdot \left(\begin{array}{l} \quad |\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) > 0| \,.\mathtt{IMB}(\boldsymbol{\alpha}, A_i) \\ + |\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) < 0| \,.\mathtt{IMB}(-\boldsymbol{\alpha}, A_i) \\ + |\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) = 0| \end{array} \right) \\ \cdot \left(\begin{array}{l} \quad |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) > 0| \,.\mathtt{IMB}(\boldsymbol{\beta}, A_i) \\ + |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) < 0| \,.\mathtt{IMB}(-\boldsymbol{\beta}, A_i) \\ + |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) = 0| \end{array} \right) \\ \cdot \left(\begin{array}{l} \quad |\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) > 0| \,.\, |\Omega_{A_i} v \geq 0| \\ + |\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) < 0| \,.\, |\Omega_{A_i} v \leq 0| \\ + |\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\gamma_i(\mathbf{x}) = 0| \end{array} \right) \end{array} \right]$$

Proof. Note that in this formulation the service rates μ_i are constant, so

$$\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\mathcal{T}v(\mathbf{x}) = \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} C(\mathbf{x}) + \sum_i \mu_i \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} \Gamma_i^{A_i} v(\mathbf{x}).$$

Then, we study the capability of Γ to propagate the property $\mathtt{S}_{\boldsymbol{\alpha},\mathbf{b}}$ for all A and all v. We have (see detail in section D.7),

$$\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\Gamma^{A}v(\mathbf{x}) = \begin{pmatrix} \gamma(\mathbf{x})\Delta_{\boldsymbol{\beta}}\Delta_{\boldsymbol{\alpha}}Av(\mathbf{x}) \\ +[1-\gamma(\mathbf{x})]\Delta_{\boldsymbol{\beta}}\Delta_{\boldsymbol{\alpha}}v(\mathbf{x}) \\ +[\Delta_{\boldsymbol{\beta}}\gamma(\mathbf{x})]\Delta_{\boldsymbol{\alpha}}\Omega_A v(\mathbf{x}+\boldsymbol{\beta}) \\ +[\Delta_{\boldsymbol{\alpha}}\gamma(\mathbf{x})]\Delta_{\boldsymbol{\beta}}\Omega_A v(\mathbf{x}+\boldsymbol{\alpha}) \\ +[\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}\gamma(\mathbf{x})].\Omega_A v(\mathbf{x}+\boldsymbol{\beta}+\boldsymbol{\alpha}) \end{pmatrix}.$$

□

Note that, if the differentiation in the direction $\mathbf{b}$ has no effect on the service rates ($\Delta_{\mathbf{b}}\gamma_i(\mathbf{x}) = 0\ \forall i$), the Theorem 6.7.3 reduces to an equivalent of the Theorem 6.5.1,

$$|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}C \geq 0| \bigotimes_{i=1}^{l} \begin{bmatrix} |A_i \to \mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}| \,.\, |\gamma_i(\mathbf{x}) \in [0,1]| \\ .\begin{pmatrix} |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) > 0| \,.\, |\Delta_{\boldsymbol{\beta}}\Omega_{A_i}v \geq 0| \\ + |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) < 0| \,.\, |\Delta_{\boldsymbol{\beta}}\Omega_{A_i}v \leq 0| \\ + |\Delta_{\boldsymbol{\alpha}}\gamma_i(\mathbf{x}) = 0| \end{pmatrix} \end{bmatrix}.$$

With the Theorem 6.7.3, we can extend our results about qualitative sensitivity analysis and some results from the literature to problem with service rate function of the state of the system. For instance, we want to know the condition for the Gamma operator on client arrival operator $\Gamma^{T^t_{\mathbf{e}_i}}$ to propagate the convexity in direction $\mathbf{e}_i$ ($\mathtt{C}_{\mathbf{e}_i}$). Knowing that $\mathtt{IMB}(\mathbf{e}_i, T^t_{\mathbf{e}_i})$ is *true* (see Table 6.5) and $T^t_{\mathbf{e}_i}$ propagates $\mathtt{C}_{\mathbf{e}_i}$ without condition, we use the Theorem 6.7.3 (with $l = 1$ and $C = 0$),

$$\begin{gathered} \left|T^t_{\mathbf{e}_i} \to \mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}\right| . |\gamma_i(\mathbf{x}) \in [0,1]| \\ .\begin{pmatrix} |\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) > 0| \,.true \\ + |\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) < 0| \,.false \\ + |\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) = 0| \end{pmatrix} . \begin{pmatrix} |\Delta_{\mathbf{e}_i}\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) > 0| \,.\, \left|\Omega_{T^t_{\mathbf{e}_i}}v \geq 0\right| \\ + |\Delta_{\mathbf{e}_i}\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) < 0| \,.\, \left|\Omega_{T^t_{\mathbf{e}_i}}v \leq 0\right| \\ + |\Delta_{\mathbf{e}_i}\Delta_{\mathbf{e}_i}\gamma_i(\mathbf{x}) = 0| \end{pmatrix}. \end{gathered}$$

So $\Gamma^{T^t_{\mathbf{e}_i}}$ propagates convexity for all linear increasing function γ ($\Delta_{\mathbf{e}_i}\gamma = cte$).

Note that the value $\Omega_{T^t_{\mathbf{e}_i}}v(\mathbf{x})$ is equal to $\Delta_{\mathbf{e}_i}v(x)$, so the function γ can be increasing and convex (resp. concave) in $\mathbf{e}_i$ if v is increasing (resp. decreasing) in $\mathbf{e}_i$.

6.8 Conclusion

This chapter provides a general framework to study the effect of the system parameters on a multidimensional queueing control problem. We decompose a generic model with three types of operators, the choice operators, the translation operators, and, in an extension, the gamma operators. We prove sufficient conditions which imply propagation of supermodularities in directions useful to have monotonic effects on the optimal policy. These conditions are expressed using Boolean equations. This formulation is chosen in order to automate future proofs on the structure and the sensitivity of the optimal policies. To develop the results of this chapter, we may consider the largest sufficient condition for T^c and T^t to propagate $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ with any $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ (not necessary in $\mathbb{A}$) and the largest sufficient condition to have $\Omega_{T^t_{\mathbf{a}}} v$ and $\Omega_{T^c_{\mathbf{a}}} v$ positive or negative. A study to find the necessary conditions in all our results may also be undertaken.

Chapitre 7

Conclusion

Les chaînes logistiques comportent de plus en plus de retours de produits. Actuellement, les causes de ces retours sont diverses, par exemple un gain économique, un argument marketing ou encore une obligation législative. Dans le futur il semble inéluctable que ces flux augmenteront dans le but de préserver les ressources naturelles limitées de notre planète.

Bien que très étudiés dans la littérature, les problèmes de gestion de flux multidimensionnels avec des retours ne considèrent pas l'impact de la capacité de production. Nos travaux s'inscrivent dans cette démarche. Nous nous plaçons dans un contexte où la capacité de production est limitée et nous considérons un problème opérationnel de gestion des stocks et de la production intégrant des flux de retours, l'objectif étant de piloter les flux de retours et de nouveaux produits de façon à satisfaire au mieux la demande et minimiser l'encours.

A partir d'un exemple général (figure 1.3, chapitre 1) prenant en compte l'acceptation des retours, leurs différentes réutilisations possibles et la coordination des flux de retours et de production traditionnel, nous déclinons plusieurs cas particuliers dans les chapitres 3 à 5. Ainsi nous modélisons trois problèmes de production et de stockage à temps continu, avec des capacités de production limitées, des délais aléatoires et des coûts linéaires. Dans le chapitre 3, nous prenons en compte la probabilité qu'un produit puisse être réutilisé comme produit fini ou seulement comme produit semi-fini (par partie). Dans le chapitre 4, nous présentons un problème où la réutilisation d'un retour comme produit fini nécessite une étape de remise à neuf. Enfin, dans le chapitre 5 nous modélisons un système où les clients préviennent à l'avance du renvoi potentiel de leurs produits.

Les principales contributions de ce document sont la modélisation des capacités de production dans des systèmes multidimensionnels avec des retours, la détermination des politiques optimales de production associées, et l'étude de politiques heuristiques pour ces systèmes (pertinence de leurs structures, et performances relatives à l'optimal). Notons que dans de nombreux cas de la littérature, les auteurs se limitent à utiliser des politiques heuristiques.

D'une façon générale, la relaxation de certaines hypothèses prises dans ce document rendrait nos modèles plus réalistes. Détaillons les principales pistes de recherche :

- Développer la modélisation de la chaîne logistique : l'exemple général présenté en figure 1.3 avec acceptation, test, et démontage/remise à neuf n'est pas traité dans ce document. Seul des cas particuliers de celui-ci sont traités. Nous pourrions par exemple combiner les modèles des chapitres 3 et 4 (i.e. ajouter une étape de production au modèle de remise à neuf). Ce modèle serait numériquement acceptable (3 dimensions) et nous pourrions peut-être démontrer des résultats de structure sur la politique optimale.

- Différencier les produits neufs et les produits remis à neuf : nous pourrions par exemple considérer deux classes de clients, la première préférant des produits neufs et la seconde des produits remis à neuf, moins chers. Une étude de tarification pourrait aussi être menée pour inciter les clients à acheter des produits neufs ou remis à neuf en fonction des disponibilités.

- Considérer des tailles de lot : dans nos modèles nous considérons toujours des approvisionnements (production/remise à neuf) unitaires. Grâce aux résultats de la section 6.7.1, nous pouvons étendre nos résultats sur les structures de politiques optimales à des approvisionnements par lots variables avec des coûts linéaires (produire 0 pièce : 0\$, 1 pièce : 1\$, ..., N pièces : N\$). Une étude de l'impact de ce changement pourrait être menée. Cependant, introduire un coût fixe par production et limiter les choix à un ensemble (0 pièce pour 0\$, 10 pièces : 15\$ ou 20 pièces : 25\$) semblerait plus réaliste.

- Introduire des coûts non linéaires : nous considérons uniquement des coûts linéaires, plusieurs autres types de coût pourraient être introduits : un coût fixe par lot, un coût fixe à la mise en marche du serveur, un coût de stockage non linéaire ou encore des économies d'échelles sur l'utilisation des ressources (1 serveur en marche : 1\$/h, 2 serveurs 1.5\$/h).

- Considérer des temps de production et des processus d'arrivée plus généraux : la formalisation en MDP devant être sans mémoire, nous sommes contraints d'utiliser des serveurs exponentiels et des processus de Poisson, cependant nous pourrions étudier l'impact de la variabilité sur nos résultats en utilisant des serveurs de temps Erlang (combinaisons de serveurs exponentiels) ou encore en évaluant la performance de nos modèles à l'aide de simulations (tirages aléatoires).

- Déterminer des approximations pour les paramètres des politiques. Certaines politiques heuristiques de ce document comportent des paramètres qu'il convient d'optimiser (nous utilisons un algorithme pas à pas en supposant l'unimodularité des tirages). Cette optimisation peut être très longue, c'est pourquoi il semble intéressant de les approximer.

Ces pistes de recherches mettent en avant le caractère restrictif des hypothèses prises dans ce document. Rappelons que la principale contrainte liée à la modélisation en MDP est la taille de l'espace d'état. Ainsi nous considérons généralement des systèmes avec au maximum deux dimensions, à part dans des cas particuliers du chapitre 4. Toutes modifications ajoutant des dimensions au problème (coûts fixes, préemption interdite, stocks supplémentaires, lois Erlang, etc...) limiteraient certainement les possibilités de caractérisation de politique optimale et la résolution numérique serait bien plus longue. Ceci étant, le caractère contraignant du modèle n'enlève rien à l'intérêt que peut avoir la relaxation de ces hypothèses.

Enfin, nous nous servons dans tout ce document d'outils permettant la caractérisation des politiques optimales. Les travaux présentés dans le chapitre 6 visent à développer ces outils. Nous présentons un cadre général pour étudier l'effet des paramètres d'un système formulé en MDP sur la politique optimale de celui-ci. Nos contributions dans ce chapitre sont diverses, tant d'un point de vue de la généricité de la formulation adoptée que des théorèmes présentés. De nombreuses pistes de recherche sont présentées. Pour certaines d'entre elles, les démonstrations associées sont très longues, ce qui constitue une limitation forte pour ces perspectives. C'est pourquoi l'automatisation de ces preuves semble être une approche intéressante pour les traiter.

En conclusion, nous avons présenté dans cette thèse des modèles multidimensionnels de gestion de stock avec des retours de produits. Nous avons mis en évidence que le contrôle optimal d'un système de production apporte un gain important dans certain cas et que la capacité de production a une incidence forte sur les politiques optimales et les performances des systèmes étudiées. C'est pourquoi nous espérons avoir contribué à améliorer la compréhension des systèmes comportant des flux de retours, permettant ainsi une meilleure intégration et un développement de ces flux.

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Appendix A

Chapter 3

A.1 Reduction to a single stage problem

A first situation where the two-stage problem reduces to the single stage problem is when the holding cost at the first stage is null ($h_1 = 0$). In this case, the following policy is optimal :

- Station M_1 : Produce all the time.
- Station M_2 : Produce if and only if $x_2 < S_2$. The base-stock level S_2 is chosen as the optimal base-stock level of a single-stage problem with parameters $(\lambda, \mu := \mu_2, h := h_2, b := b, \delta := \delta_2, \alpha)$.

A second situation is when the second station can produce items instantaneously ($\mu_2 = \infty$). In this situation, we must distinguish three cases. If $h_1 \geq h_2$, then the optimal policy is :

- Station M_1 : Produce if and only if $x_1 + x_2 < S_1$. The echelon base-stock level S_1 is chosen as the optimal base-stock level of a single-stage problem with parameters $(\lambda, \mu := \mu_1, h := h_2, b, \delta := \delta_1 + \delta_2, \alpha)$.
- Station M_2 : Produce whenever possible.

If $h_1 < h_2$ and $\delta_2 = 0$, then the optimal policy is :

- Station M_1 : Produce if and only if $x_1 + x_2 < S_1'$. The echelon base-stock level S_1' is chosen as the optimal base-stock level of a single-stage problem with parameters $(\lambda, \mu := \mu_1, h := h_1, b, \delta := \delta_1, \alpha)$.
- Station M_2 : Operate in a make-to-order fashion, i.e. produce if and only if $x_2 < 0$.

If $h_1 < h_2$ and $\delta_2 > 0$, the two-stage problem does not reduce to a single-stage problem.

A.2 Proof of Lemma 3.3.3

We have

$$\begin{aligned}
v_\alpha^S(S) &= E\left[\int_0^\infty e^{-\alpha t} c(X(t))dt|X(0)=S\right] \\
&= \int_0^\infty e^{-\alpha t} E\left[c(X(t))|X(0)=S\right] dt \\
&= \int_0^\infty e^{-\alpha t} \left\{\sum_y c(y) P[X(t)=y|X(0)=S]\right\} dt \\
&= \sum_y \left\{c(y) \int_0^\infty e^{-\alpha t} P[X(t)=y|X(0)=S] dt\right\}.
\end{aligned}$$

Let $p_y(t) = P[X(t)=y|X(0)=S]$ be the transient probability to be in state y at time t, when the initial state is S. Let $\tilde{p}_y(\alpha)$ be the Laplace transform of $p_y(t)$:

$$\tilde{p}_y(\alpha) = \int_0^\infty e^{-\alpha t} p_y(t) dt.$$

Then

$$\begin{aligned}
v_\alpha^S(S) &= \sum_y c(y) \int_0^\infty e^{-\alpha t} p_y(t) \\
&= \sum_y c(y) \tilde{p}_y(\alpha).
\end{aligned}$$

In order to compute $\tilde{p}_y(\alpha)$, we write the differential equations on transient probabilities:

$$\begin{aligned}
p_y' &= -(\lambda+\delta)p_y + \lambda p_{y+1} + \delta p_{y-1} && \text{if } y > S, \\
p_y' &= -(\lambda+\delta)p_y + \lambda p_{y+1} + (\mu+\delta) p_{y-1} && \text{if } y = S, \\
p_y' &= -(\lambda+\mu+\delta)p_y + \lambda p_{y+1} + (\mu+\delta) p_{y-1} && \text{if } y < S,
\end{aligned}$$

where $p_y'(t)$ denotes the first derivative of $p_y(t)$. By taking the Laplace transform of the previous set of differential equations, we obtain

$$(\alpha+\lambda+\delta)\tilde{p}_y = \lambda\tilde{p}_{y+1} + \delta\tilde{p}_{y-1} \quad \text{if } y > S, \tag{A.1}$$
$$(\alpha+\lambda+\delta)\tilde{p}_y = 1 + \lambda\tilde{p}_{y+1} + (\mu+\delta)\tilde{p}_{y-1} \quad \text{if } y = S, \tag{A.2}$$
$$(\alpha+\lambda+\mu+\delta)\tilde{p}_y = \lambda\tilde{p}_{y+1} + (\mu+\delta)\tilde{p}_{y-1} \quad \text{if } y < S. \tag{A.3}$$

(A.1) and (A.3) are second-order linear recurrence and have the following solutions

$$\begin{cases}
\tilde{p}_y(\alpha) = A_1\alpha_1^{S-y} + B_1\beta_1^{S-y} & \text{if } y \le S, \\
\tilde{p}_y(\alpha) = A_2\alpha_2^{y-S} + B_2\beta_2^{y-S} & \text{if } y \ge S,
\end{cases} \tag{A.4}$$

where α_1, β_1 are the roots of the characteristic equation

$$(\mu+\delta)x^2 - (\alpha+\lambda+\mu+\delta)x + \lambda = 0, \tag{A.5}$$

and α_2, β_2 are the roots of another characteristic equation

$$\lambda x^2 - (\alpha+\delta+\lambda)x + \delta = 0. \tag{A.6}$$

Solving quadratic equations (A.5) and (A.6) gives

$$\begin{matrix}\alpha_1\\ \beta_1\end{matrix} = \frac{\alpha+\lambda+\delta+\mu \pm \sqrt{(\alpha+\lambda+\mu+\delta)^2 - 4\lambda(\mu+\delta)}}{2(\mu+\delta)},$$

and

$$\begin{matrix}\alpha_2\\ \beta_2\end{matrix} = \frac{\alpha+\lambda+\delta \pm \sqrt{(\alpha+\lambda+\delta)^2 - 4\lambda\delta}}{2\lambda}.$$

We observe that $\alpha_i > 1$ and $0 < \beta_i < 1$ for $i = 1, 2$.

On one hand, we have $\sum_y \tilde{p}_y(\alpha) = 1/\alpha$ since $\sum_y p_y(t) = 1$. On the other hand, we have

$$\sum_y \tilde{p}_y(\alpha) = \sum_{y=-\infty}^{S} (A_1\alpha_1^{S-y} + B_1\beta_1^{S-y}) + \sum_{y=S+1}^{+\infty} (A_2\alpha_2^{y-S} + B_2\beta_2^{y-S}).$$

The convergence of $\sum_y \tilde{p}_y(\alpha)$ implies that $A_1 = A_2 = 0$.

Using (A.4) when $y = S$ gives $\tilde{p}_S(\alpha) = B_1 = B_2 = B$ and we get

$$\begin{cases} \tilde{p}_y(\alpha) = B\beta_1^{S-y} & \text{if } y \le S, \\ \tilde{p}_y(\alpha) = B\beta_2^{y-S} & \text{if } y \ge S. \end{cases}$$

Then (A.2) gives

$$B = \frac{1}{\alpha+\lambda+\delta-(\mu+\delta)\beta_1 - \lambda\beta_2}.$$

As β_1, β_2 respectively satisfy the quadratic equations (A.5) and (A.6), we have

$$\lambda - (\mu+\delta)\beta_1 = \alpha\frac{\beta_1}{1-\beta_1}, \quad \delta - \lambda\beta_2 = \alpha\frac{\beta_2}{1-\beta_2}.$$

Then

$$B = \frac{1}{\alpha + \alpha\frac{\beta_1}{1-\beta_1} + \alpha\frac{\beta_2}{1-\beta_2}} = \frac{1}{\alpha}\frac{(1-\beta_1)(1-\beta_2)}{1-\beta_1\beta_2}.$$

Finally, for $S \geq 0$ we obtain

$$\begin{aligned}
v_\alpha^S(S) &= -b \sum_{i=-\infty}^{0} i\tilde{p}_i(\alpha) + h \sum_{x=0}^{+\infty} i\tilde{p}_i(\alpha) \\
&= -b \sum_{i=-\infty}^{0} i\beta_1^{S-i} B + h \sum_{i=0}^{S} i\beta_1^{S-i} B + h \sum_{i=S+1}^{+\infty} i\beta_2^{i-S} B \\
&= \frac{h}{\alpha} \left\{ S + \alpha B \left[\frac{\beta_1}{(1-\beta_1)^2} \left(-1 + \frac{h+b}{h} \beta_1^S \right) + \frac{\beta_2}{(1-\beta_2)^2} \right] \right\},
\end{aligned}$$

and for $S \leq 0$ we obtain

$$v_\alpha^S(S) = \frac{b}{\alpha} \left\{ -S + \alpha B \left[\frac{\beta_2}{(1-\beta_2)^2} \left(-1 + \frac{b+h}{b} \beta_2^{-S} \right) + \frac{\beta_1}{(1-\beta_1)^2} \right] \right\}.$$

A.3 Computational procedure

To compute the optimal policy, we truncate the state space in three directions. Let Γ_1 and Γ_2^+ two positive integers and Γ_2^- a negative integer :

$$0 \leq x_1 \leq \Gamma_1 \text{ and } \Gamma_2^- \leq x_2 \leq \Gamma_2^+.$$

We can then apply a value iteration algorithm (Puterman, 1994) to this truncated state. We increase the state space until the average cost is no more sensitive to the truncation level.

In order to evaluate a heuristic policy with parameters (S_1, S_2), we apply the same procedure except that we must change the production operators. For all heuristics, the control is similar at stage 2 and operator T_2 is be replaced by

$$\tilde{T}_2 v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) & \text{if } x_1 > 0, x_2 < S_2, \\ v(\mathbf{x}) & \text{else.} \end{cases}$$

At stage 1, the control policy depends on the policy. For the half-optimal policy, T_2 remains unchanged. For base-stock policy, Kanban policy and fixed-buffer policy, operator T_2 is respectively replaced by

$$T_1^{BSE}v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{e}_1) & \text{if } x_1 + x_2 < S_1, \\ v(\mathbf{x}) & \text{otherwise,} \end{cases}$$

$$T_1^{KB}v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{e}_1) & \text{if } x_1 + x_2^+ < S_1, \\ v(\mathbf{x}) & \text{otherwise,} \end{cases}$$

$$T_1^{FB}v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{e}_1) & \text{if } x_1 < S_1, \\ v(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Denote by $C^{\pi}(S_1, S_2)$ the average cost of policy π, with π = Kanban, fixed-buffer, base-stock. For each class of policies, we want to find the parameters $S_1^\star, S_2^\star$ minimizing $C(S_1, S_2)$. This optimization problem is a non linear problem with integer variables that might be long to solve since evaluating a given policy might already be time consuming. Therefore, we make the plausible assumption that the function $C(S_1, S_2)$ is unimodal. This assumption has been checked on several instances. Based on the unimodularity assumption, we can solve efficiently the problem with the maximal gradient with constant step method. This method is very efficient here because we can start the optimization with an approximate value of $S_1^\star$ and $S_2^\star$, resulting from the calculation of the optimal policy.

Appendix B

Chapter 4

B.1 Proof of Theorem 4.4.1

Because $\mathcal{T}$ is a contraction mapping, the fixed point theorem ensures that $v_{n+1} = \mathcal{T}v_n$ converges to the optimal value function v^*, which is the unique solution of the optimality equation $v^\star = Tv^\star$ (Puterman, 1994). Moreover, $\mathcal{T}$ is a convex combination of cost function (C) and event operators denoted $T_{A(1)}$, $T_{CJ(1,2)}$, and $T_{A(2)}$ in Koole (1998) for respectively T_e, T_r, and T_m. Koole (1998) proves that this operators propagates the properties of supermodularity and superconvexity. If we take $v_0(x_1, x_2) = 0 \ \forall (x_1, x_2)$, it is clear that $v_0 \in V$ and then via induction $v^\star \in V$.

As $v^* \in V$, supermodularity and superconvexity ensure that the three switching curves are well defined. For instance, convexity in x_2 ensures that we can define the manufacturing threshold $S_m(x_1) = \min[x_1 | v(x_1, x_2+1) - v(x_1, x_2) + c^m > 0]$. The monotonicity results for the switching curves are also implied by the fact that $v^* \in V$. For instance, supermodularity ensures that $S_m(x_1 + 1) \leq S_m(x_1)$.

B.2 Proof of Theorem 4.4.2

We want to prove by induction that $\Delta_{\mathbf{e_2}-\mathbf{e_1}} v^\star(x_1, x_2) = v^\star(x_1 - 1, x_2 + 1) - v^\star(x_1, x_2) \leq 0$. Let v be the value function such that $\Delta_{\mathbf{e_2}-\mathbf{e_1}} v \leq 0$. We want to prove that $\Delta_{\mathbf{e_2}-\mathbf{e_1}} \mathcal{T}v \leq 0$.

Koole (2006) proves that operators T_e, T_r, and T_m propagate the property $\Delta_{\mathbf{e_2}-\mathbf{e_1}} v \leq 0$. In addition, we have

$$\Delta_{\mathbf{e_2}-\mathbf{e_1}} c(x_1, x_2) = c(x_1 - 1, x_2 + 1) - c(x_1, x_2) = \begin{cases} h_1 - h_2 \leq 0 & \text{if } x_2 \geq 0, \\ -h_1 - b \leq 0 & \text{else.} \end{cases}$$

So we can conclude with (4.1) that $\Delta_{\mathbf{e_2}-\mathbf{e_1}} \mathcal{T}v \leq 0$ and by induction that $\Delta_{\mathbf{e_2}-\mathbf{e_1}} v^\star \leq 0$.

B.3 System simplification

The set of parameters for our problem is large:

$$\{\delta, \mu_r, \mu_m, \lambda, h_1, h_2, b, c^a, c^b, c^m, c^r, \alpha\}.$$

We want to simplify our model by reducing the number of parameters for the average cost criteria.

Let $f_m(\pi)$, $f_r(\pi)$, $f_a(\pi)$ and $f_d(\pi)$ be respectively the average flow of product from manufacturing, from remanufacturing, of accepted returns and of rejected returns. The average cost $C(\pi) = f_m(\pi)c^m + f_d(\pi)c^b + f_a(\pi)c^a + f_r(\pi)c^r + H(\pi)$, with $H(\pi)$ the average cost of storage and backlogs. We know that $f_r(\pi) = f_a(\pi) = \delta - f_d(\pi)$ and $f_m(\pi) = \lambda - \delta + f_d(\pi)$ so $C(\pi) = f_d(\pi)(c^b - c^a - c^r + c^m) + \delta(c^a + c^r - c^m) + \lambda c^m + H(\pi)$.

Without loss of generality we can set $c^b = c^r = c^m = 0$ and create a relative acceptance cost $c = c^b - c^a - c^r + c^m$. Note that c can be negative. The actual system has the same average cost optimal policy and its average cost is the same with a constant offset $\delta(c^a + c^r - c^m) + \lambda c^m$. This simplification does not hold for discounted cost, however we will assume $c^b = c^r = c^m = 0$ to reduce the number of parameters for the discounted case. This type of approximation is standard in the literature (Veatch and Wein, 1994).

Moreover, we assume $\lambda = 1$ and $h_1 = 1$ without loss of generality. Then, the initial set of parameters

$$\{\delta, \mu_r, \mu_m, \lambda, h_1, h_2, b, c^a, c^b, c^m, c^r, \alpha\}$$

reduces to

$$\{\delta', \mu'_r, \mu'_m, h'_2, b', c, \alpha'\}.$$

In the numerical study, we can not consider negative values of c because the expected discounted/average cost could be 0 or negative. In this case the computation is longer because the convergence of the fixed point algorithm is longer when the cost is close to 0. Moreover the comparison between policies is difficult when some of the costs are positives and others negatives. So we set $c^m = 5$, $c^a = c^r = 0$ and $c^b \in \{0, 5, 10\}$.

B.4 Computational procedure

To compute the average cost when using the different policies, we truncate the state space in three directions. Let M_1, M_2^+ and M_2^- denote three integers with $0 \leq x_1 \leq M_1$ and $M_2^- \leq x_2 \leq M_2^+$. We apply a value iteration algorithm (Puterman, 1994) to this truncated state space and we increase the state space until the discounted/average cost is no more sensitive to the truncation level with 5 digits accuracy.

For the computation with a heuristic strategy, the MDP formulation given in Section 4.3 is adapted. For instance, the operator of entrance of returns with strategy $x_1 + x_2$

becomes:

$$\tilde{T}_e v(x_1, x_2) = \begin{cases} v(x_1 + 1, x_2) & \text{if } x_1 + x_2 < z_e \\ v(x_1, x_2) & \text{else} \end{cases}$$

For each policy π, we want to find the set of values for the parameters $\{s_1, \dots, s_n\}$ minimizing the cost $C_\alpha^\pi(s_1, \dots, s_n)$ (with discount rate $\alpha \in \mathbb{R}^+$). This optimization problem is a difficult non linear problem with integer variables. We make the assumption that $C_\alpha^\pi(s_1, \dots, s_n)$ is unimodal and we look for the minimum with the maximal gradient with constant step method. The assumption of unimodularity is valid for all numerically studied instances.

B.5 Direct reuse

Case 1 : $c^a < c^m + c^b$

Now we can again restrict the optimization search to (S_m, S_a) policies with $S_m \geq S_a$ (see (4.2)). We fix $w := S_m - S_a$ ($w \geq 0$) and we look for $S_m^\star(w)$ the optimal base stock level for a given w. We notice that the stochastic process $N(t) = S_m - X(t)$ is independent of S_m if w is fixed. In the following we simplify the notation $N(t)$ (resp. $X(t)$) by N (resp. X). Then the stationary distribution of N is given by

$$P(N = 0) = \frac{(1 - \rho_r)(1 - \rho)}{1 - \rho_r - \rho^w(\rho - \rho_r)},$$

$$P(N = n) = \begin{cases} \rho^n \; P(N = 0) & \text{if } 0 \leq n \leq w, \\ \rho_r^{n-w} \rho^w P(N = 0) & \text{if } n \geq w. \end{cases}$$

The stationary distribution of N does not depend on S_m if w is fixed. To prove that $S_m^\star \geq 0$, with $P(N) = P_{S_m}(X) = P_{S_m+1}(X + 1)$, we first assume $S_m < 0$:

$$\begin{aligned}
&c(w, S_m + 1) - c(w, S_m) \\
&= b(E_{S_m+1}[X^-] - E_{S_m}[X^-]) \\
&\quad + c^m \mu_m (P_{S_m+1}(X < S_m + 1) - P_{S_m}(X < S_m)) \\
&\quad + c^a \delta (P_{S_m+1}(X < S_a + 1) - P_{S_m}(X < S_a)) \\
&\quad + c^b \delta (P_{S_m+1}(X \geq S_a + 1) - P_{S_m}(X \geq S_a)) \\
&= b(E_{S_m+1}[X^-] - E_{S_m}[X^-]) \\
&= -b \left[\textstyle\sum_{-\infty}^{S_m+1} X P_{S_m+1}(X) - \sum_{-\infty}^{S_m} X P_{S_m}(X) \right] \\
&= -b \left[\textstyle\sum_{-\infty}^{S_m} (X + 1) P_{S_m}(X) - \sum_{-\infty}^{S_m} X P_{S_m}(X) \right] \\
&= -b \left[1 - P_{S_m}(X) \right].
\end{aligned}$$

The average cost for threshold S_m is bigger than the average cost for threshold $S_m + 1$, so $S_m^\star \geq 0$.

The average cost is given by

$$\begin{aligned}
c(w, S_m) &= hE[(S_m - N)^+] + bE[(S_m - N)^-] \\
&\quad + c^m \mu_m P(X < S_m) + c^a \delta P(X < S_a) \\
&\quad + c^b \delta P(X \geq S_a) \\
&= hE[(S_m - N)^+] + bE[(S_m - N)^-] \\
&\quad + c^m \mu_m P(N > 0) \\
&\quad + \delta((c^a - c^b) P(N > w) + c^b)
\end{aligned}$$

When w is fixed, $P(N > w)$ and $P(N > 0)$ do not depend on S_m so we have to solve the following minimization problem:

$$\min_s \left\{ hE[(s - N)^+] + bE[(s - N)^-] \right\}.$$

This minimization problem is equivalent to a newsboy problem where the order quantity is s, the stochastic demand is N, the shortage cost is b and the holding cost is h. We have the following known result (see for instance Porteus (2002)) :

$$\begin{aligned}
S_m^*(w) &= \operatorname*{argmax}_s \left\{ hE[(s - N)^+] + bE[(s - N)^-] \right\} \\
&= \min \left\{ s | F(s) \geq \frac{b}{b + h} \right\}
\end{aligned}$$

where F is the cumulative distribution of N. The cumulative distribution of N is easy to compute:

$$\begin{aligned}
F(S_m) &= \sum_{n=-\infty}^{S_m} P(N = n) \\
&= \frac{1 - \rho^w}{1 - \rho} P(0) + \frac{1 - \rho_r^{S_m - w + 1}}{1 - \rho_r} \rho^w P(0).
\end{aligned}$$

We are now able to compute $S_m^*(w)$ for all w. Moreover if we assume that $S_m^*(w)$ is non-decreasing in w, we should have

$$S_m^*(w = 0) \leq S_m^* \leq S_m^*(w = \infty)$$

with $S_m^*(w = 0)$ (resp. $S_m^*(w = \infty)$) the optimal base-stock level of an $M/M/1$ make to-stock queue with backorders, no return and utilization rate ρ_r (resp. ρ). From Veatch and Wein (1996), we have:

$$S_m^*(w=0) = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \rho_r} \right\rfloor \quad S_m^*(w=\infty) = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \rho} \right\rfloor.$$

Case 2: $c^a > c^m + c^b$

The same approach can be used by letting $w := S_a - S_m$, $N(t) = S_a - X(t)$. Then we have

$$P(N=0) = \frac{(1-\rho_r)(1-1/\rho)}{1-\rho_r-\rho^{-w}(1/\rho-\rho_r)},$$
$$P(N=n) = \begin{cases} \rho^{-n}\ P(N=0) & \text{if } 0 \le n \le w, \\ \rho_r^{n-w}\rho^{-w}P(N=0) & \text{if } n \ge w. \end{cases}$$

and F the cumulative distribution of N is given by

$$F(S_a) = \frac{1-\rho^{-w}}{1-1/\rho}P(N=0) + \frac{1-\rho_r^{S_a-w+1}}{1-\rho_r}\rho^w P(N=0)$$

and

$$S_a^*(w) = \min\left\{ s | F(s) \ge \frac{b}{b+h} \right\}.$$

Case 3: $c^a = c^m + c^b$

Restricting the optimization to (S_m, S_m) policies (i.e. $w=0$), we have immediately $P(N=0) = 1-\rho_r$ and $F(S_m) = 1-\rho_r^{S_m+1}$. Then we have

$$F(S_m) = \frac{b}{h+b} \Leftrightarrow S_m = \frac{\ln \frac{h}{h+b}}{\ln \rho_r} - 1$$

and

$$S_a^* = S_m^* = \left\lfloor \frac{\ln \frac{h}{h+b}}{\ln \rho_r} \right\rfloor$$

which is consistent with Veatch and Wein (1996).

B.6 Disposal option

B.6.1 Proof of the theorem 4.7.1

First we prove that the dispose decision occurs only with an event (an arrival of a return, a demand of a client or an arrival of new product). Suppose two consecutive events at time t_1 and t_2 with respective cost E_1 and E_2. An dispose decision occurs at time $t_d \in [t_1, t_2]$ with a cost c^d. With the cost function $C(x) = hx^+ + bx^-$, between this two events the total discounted cost $C_\alpha^{[t_1,t_2]}$ is given by:

$$\begin{aligned} C_\alpha^{[t_1,t_2]} &= \int_{t_1}^{t_2} e^{-\alpha t} C(x(t)) + \sum_{i=1}^{2} e^{-\alpha t_i} E_i + e^{-\alpha t_d} k c^d \\ &= \int_{t_1}^{t_d} e^{-\alpha t} C(x(t_1)) + \int_{t_d}^{t_2} e^{-\alpha t} (C(x(t_1)) - kh) + \sum_{i=1}^{2} e^{-\alpha t_i} E_i + e^{-\alpha t_d} k c^d \\ &= cte + e^{-\alpha t_d} \left(k c^d - \frac{h}{\alpha} \right) \end{aligned}$$

Because $\operatorname{argmin}_{t_d} \left\{ C_\alpha^{[t_1,t_2]} \right\} \in \{t_1, t_2\}$, the dispose decision occurs only with events.

For time $t > 0$ the MDP formulation is

$$Tv(x) = \frac{1}{\tau} \left[C(x) + \lambda \left(\prod_0^\infty T_d \right) v(x-1) + \mu_m \left(\prod_0^\infty T_d \right) T_m v(x) + \delta \left(\prod_0^\infty T_d \right) T_a v(x) \right]$$

with the uniformisation rate $\tau = \alpha + \lambda + \mu_m + \delta$ and

- $T_d = \begin{cases} \min\{v(x), v(x-1) + c^d\} \text{ if } x > 0, \\ v(x) \text{ otherwise }, \end{cases}$ the disposal operator,
- $T_a = \min\{v(x) + c^b, v(x+1) + c^a\}$ the acceptance operator, and
- $T_m = \min\{v(x), v(x+1) + c^m\}$ the manufacturing operator.

Note that $(\prod_0^\infty T_d) = \min_{k \in \{0,x^+\}} \{v(x-k) + kc^d\}$ is an operator of disposal by batch. Because T_d, T_a, T_m, and C propagate convexity (Koole, 1998), the unique solution of the equation $v^\star = Tv^\star$ is convex. So the structure of the optimal policy is a three-threshold (z_a, z_d, z_m) policy.

Let v^- be the discounted value function on period $[0, \infty[$. At $t = 0$, if $x_0 > z_d$, the quantity $x_0 - z_d$ is disposed so $v^-(y) = v(x_0) + \max\{x_0 - z_d, 0\}$. So v^- is convex too and its three thresholds are the same than v. In the following we focus only on system with $x_0 \leq z_d$.

Now we want to simplify the MDP formulation. First $\prod_0^\infty T_d T_m v = T_m v$, because

$$\begin{aligned} T_d T_m v(x) &= \min \begin{cases} v(x), v(x+1), v(x) + c^d + c^m v(x-1) + c^d, & \text{if } x > 0, \\ v(x), v(x+1), v(x) + c^d + c^m & \text{otherwise.} \end{cases} \\ &= T_m v(x) \text{ because } c^d + c^m \geq 0, \end{aligned}$$

In the same way, we have $\prod_0^\infty T_d v(x-1) = v(x-1)$. Then $\prod_0^\infty T_d T_a v(x) = T_d T_a v(x)$ because,

- $x < z_d$: $T_d T_a v(x) = T_a v(x)$ because in the worst case $T_a v(x) = v(z_d) + c^a$.
- $x = z_d$: $T_d T_a v(z_d) = v(z_d) + \min\{c^b, c^a + c^d\}$, so $T_d T_d T_a v(x) = T_d T_a v(x)$.

- $x > z_d$: this case is not possible $\forall t > 0$.

So, the MDP formulation reduces to

$$Tv(x) = \frac{1}{\tau}\left[C(x) + \lambda v(x-1) + \mu_m T_m v(x) + \delta T_d T_a v(x)\right],$$

and the disposal occurs one by one, only on return event.

Moreover, $T_d T_a v(x) = \min\{v(x)+c^b, v(x+1)+c^a, v(x)+c^a+c^d\}$. So, when $c^d+c^a \le c^b$ (resp. $\ge c^b$), the rejection (resp. disposal) option is never used.

B.6.2 Proof of the theorem 4.7.2

Proof. As in single stage problem (see section B.6.1), the dispose decision occurs only with events, and products are disposed one by one. On period $]0,\infty[$, with

- the state space $\mathbf{x} = (x_1, x_2) \in \mathbb{N} \times \mathbb{Z}$.
- the uniformisation rate $\tau = \alpha + \lambda + \mu_m + \mu_r + \delta$,
- the cost function $C(\mathbf{x}) = h_1 x_1 + h_2 x_2^+ + b x_2^-$,
- the disposal operator $T_d = \begin{cases} \min\{v(\mathbf{x}), v(\mathbf{x}-\mathbf{e}_1) + c^d\} \text{ if } x_1 > 0, \\ v(\mathbf{x}) \text{ otherwise }, \end{cases}$
- the acceptance operator $T_a = \min\{v(\mathbf{x}) + c^b, v(\mathbf{x}+\mathbf{e}_1) + c^a\}$,
- the manufacturing operator $T_m = \min\{v(\mathbf{x}), v(\mathbf{x}+\mathbf{e}_2) + c^m\}$, and
- the remanufacturing operator $T_r = \begin{cases} \min\{v(\mathbf{x}), v(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2) + c^r\} \text{ if } x_1 > 0, \\ v(\mathbf{x}) \text{ otherwise,} \end{cases}$

the MDP formulation is

$$Tv(\mathbf{x}) = \frac{1}{\tau}\left[\begin{array}{l} C(\mathbf{x}) + \lambda \left(\prod_0^\infty T_d\right) v(\mathbf{x}-\mathbf{e}_2) + \mu_m \left(\prod_0^\infty T_d\right) T_m v(\mathbf{x}) \\ +\mu_r \left(\prod_0^\infty T_d\right) T_r v(\mathbf{x}) + \delta \left(\prod_0^\infty T_d\right) T_a v(\mathbf{x}) \end{array}\right]$$

and can be simplified to

$$Tv(\mathbf{x}) = \frac{1}{\tau}\left[\; C(\mathbf{x}) + \lambda v(\mathbf{x}-\mathbf{e}_2) + \mu_m T_d T_m v(\mathbf{x}) + \mu_r T_r v(\mathbf{x}) + \delta T_d T_a v(\mathbf{x}) \;\right]$$

Because T_d, T_a, T_m, T_r, and C propagate supermodularity and superconvexity (Koole, 1998), the unique solution of the equation $v^\star = Tv^\star$ is supermodular and superconvex. So the structure of the optimal policy is a four-threshold $(z_a(x_2), z_d(x_2), z_m(x_1), z_r(x_1))$ policy.

Moreover $T_d T_a v(\mathbf{x}) = \min\{v(\mathbf{x}+\mathbf{e}_1)+c^a, v(\mathbf{x})+c^a+c^d, v(\mathbf{x})+c^b\}$. Then, if $c^a+c^d \le c^b$ the MDP formulation reduces to

$$Tv(\mathbf{x}) = \frac{1}{\tau}\Big[\ C(\mathbf{x}) + \lambda v(\mathbf{x}-\mathbf{e}_2) + \mu_m T_d T_m v(\mathbf{x}) + \mu_r T_r v(\mathbf{x}) + \delta(T_d v(\mathbf{x}+\mathbf{e}_1) + c^a)\ \Big]$$

and the rejection option is never used. If $c^a + c^d \ge c^b$ the MDP formulation reduces to

$$Tv(\mathbf{x}) = \frac{1}{\tau}\Big[\ C(\mathbf{x}) + \lambda v(\mathbf{x}-\mathbf{e}_2) + \mu_m T_d T_m v(\mathbf{x}) + \mu_r T_r v(\mathbf{x}) + \delta T_a v(\mathbf{x}+\mathbf{e}_1)\ \Big]$$

and the disposal option is never used with return event. $\square$

Appendix C

Chapter 5

C.1 Proof of Lemma 5.3.1

As operator T is a contraction mapping (Puterman,1994), the fixed point theorem ensures that the sequence of value functions $v_{n+1} = Tv_n$ converges to $v^\star$, for any v_0, and in particular if v_0 is the null value function which belongs to U. In the following, we show that operator T preserves conditions of U, i.e. if $v \in U$, then $Tv \in U$. By induction, we can then conclude that $v^\star \in U$.

Define $S_y = \min\left[x|\Delta v(x,y) + c^m \geq 0\right]$. Operator T_0 can be rewritten as follows.

$$T_0 v(x,y) = \begin{cases} v(x+1,y) & \text{if } x < S_y, \\ v(x,y) & \text{if } x \geq S_y. \end{cases}$$

We can then compute $\Delta T_0 v$.

$$\Delta T_0 v(x,y) = \begin{cases} \Delta v(x+1,y) & \text{if } x < S_y - 1, \\ -c^m & \text{if } x = S_y - 1, \\ \Delta v(x,y) & \text{if } x \geq S_y. \end{cases}$$

As v satisfies condition C.4 ($\Delta v(x,y) \geq -c^l$) and $c^m < c^l$, we immediately obtain that $T_0 v$ satisfies condition C.4 ($\Delta T_0 v(x,y) \geq -c^l$).

We can now prove that $T_0 v$ satisfies conditions C.1, C.2 and C.3 of U. All the inequalities below are obtained by using the assumption that $v \in U$.

$$\Delta^2 T_0 v(x,y) = \begin{cases} \Delta^2 v(x+1,y) \geq 0 & \text{if } x < S_y - 2, \\ -\Delta v(x+1,y) - c^m \geq 0 & \text{if } x = S_y - 2, \\ \Delta v(x+1,y) + c^m \geq 0 & \text{if } x = S_y - 1, \\ \Delta^2 v(x,y) & \text{if } x \geq S_y. \end{cases}$$

$$
\begin{aligned}
&\Delta T_0 v(x,y+1) - \Delta T_0 v(x,y)\\
&= \begin{cases}
\Delta v(x+1,y+1) - \Delta v(x+1,y) \geq 0 & \text{if } x < S_y - 1,\\
\Delta v(x,y+1) + c^m \geq 0 & \text{if } x = S_y - 1 \text{ and } x = S_{y+1},\\
0 \geq 0 & \text{if } x = S_y - 1 \text{ and } x = S_{y+1} - 1,\\
\Delta v(x,y+1) - \Delta v(x,y) & \text{if } x \geq S_y.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\Delta T_0 v(x+1,y) - \Delta T_0 v(x,y+1)\\
&= \begin{cases}
\Delta v(x+2,y) - \Delta v(x+1,y+1) \geq 0 & \text{if } x < S_y - 2\\
-c^m - \Delta v(x+1,y+1) \geq 0 & \text{if } x = S_y - 2 \text{ and } x = S_{y+1} - 2\\
0 & \text{if } x = S_y - 2 \text{ and } x = S_{y+1} - 1\\
\Delta v(x+1,y) - \Delta v(x,y+1) \geq 0 & \text{if } x = S_y - 1 \text{ and } x = S_{y+1}\\
\Delta v(x+1,y) + c^m \geq 0 & \text{if } x = S_y - 1 \text{ and } x = S_{y+1} - 1\\
\Delta v(x+1,y) - \Delta v(x,y+1) \geq 0 & \text{if } x \geq S_y
\end{cases}
\end{aligned}
$$

We conclude that $T_0 v$ belongs to U.

Similarly, we prove that $T_1 v$ belongs to U.

$$
\Delta T_1 v(x,y) = \begin{cases}
\Delta v(x-1,y) \geq -c^l & \text{if } x > 0,\\
-c^l \geq -c^l & \text{if } x = 0.
\end{cases}
$$

$$
\Delta^2 T_1 v(x,y) = \begin{cases}
\Delta^2 v(x-1,y) \geq 0 & \text{if } x > 0,\\
0 \geq 0 & \text{if } x = 0.
\end{cases}
$$

$$
\Delta T_1 v(x,y+1) - \Delta T_1 v(x,y) = \begin{cases}
\Delta v(x-1,y+1) - \Delta v(x+1,y) \geq 0 & \text{if } x > 0,\\
0 \geq 0 & \text{if } x = 0.
\end{cases}
$$

$$
\Delta T_1 v(x+1,y) - \Delta T_1 v(x,y+1) = \begin{cases}
\Delta v(x,y) - \Delta v(x-1,y+1) \geq 0 & \text{if } x > 0,\\
0 \geq 0 & \text{if } x = 0.
\end{cases}
$$

We also obtain that $T_2 v \in U$.

$$
\Delta T_2 v(x,y) = \begin{cases}
\Delta v(x,y+1) \geq -c^l & \text{if } y < M\\
\Delta v(x,y) \geq -c^l & \text{if } y = M
\end{cases}
$$

$$
\Delta^2 T_2 v(x,y) = \begin{cases}
\Delta^2 v(x,y+1) \geq 0 & \text{if } y < M\\
\Delta^2 v(x,y) \geq 0 & \text{if } y = M
\end{cases}
$$

$$
\Delta T_2 v(x,y+1) - \Delta T_2 v(x,y) = \begin{cases}
\Delta v(x,y+2) - \Delta v(x,y+1) \geq 0 & \text{if } y < M-1\\
0 & \text{if } y = M-1
\end{cases}
$$

$$\Delta T_2 v(x+1,y) - \Delta T_2 v(x,y+1) = \begin{cases} \Delta v(x+1,y+1) - \Delta v(x,y+2) \geq 0 & \text{if } y < M-1 \\ \Delta v(x+1,y+1) - \Delta v(x,y+1) \geq 0 & \text{if } y = M-1 \end{cases}$$

and that $T_3 v \in U$. Moreover, we prove that $\Delta T_3 v(x,y) \geq -M\,c^l$.

$$\Delta T_3 v(x,y) = \begin{cases} y\Delta v(x+1,y-1) + (M-y)\Delta v(x,y) \geq -Mc^l & \text{if } y > 0 \\ M\Delta v(x,y) \geq -Mc^l & \text{if } y = 0 \end{cases}$$

$$\Delta^2 T_3 v(x,y) = \begin{cases} y(\Delta^2 v(x+1,y-1) + c^r) + (M-y)\Delta^2 v(x,y) \geq 0 & \text{if } y > 0 \\ M\Delta^2 v(x,y) \geq 0 & \text{if } y = 0 \end{cases}$$

$$\Delta T_3 v(x,y+1) - \Delta T_3 v(x,y) = \begin{cases} y(\Delta v(x+1,y) - \Delta v(x+1,y-1)) & \\ \quad +(M-y)(\Delta v(x,y+1) - \Delta v(x,y)) & \text{if } y > 0 \\ \quad +\Delta v(x+1,y) - \Delta v(x,y+1) \geq 0 & \\ (y+1)(\Delta v(x+1,y) - \Delta v(x,y+1)) & \text{if } y = 0 \\ \quad +M(\Delta v(x,y+1) - \Delta v(x,y)) \geq 0 & \end{cases}$$

$$\Delta T_3 v(x+1,y) - \Delta T_3 v(x,y+1) = \begin{cases} (M-y-1)(\Delta v(x+1,y) - \Delta v(x,y+1)) & \text{if } y > 0 \\ \quad +y\,(\Delta v(x+2,y-1) - \Delta v(x+1,y)) \geq 0 & \\ (M-y-1)(\Delta v(x+1,y) - \Delta v(x,y+1)) \geq 0 & \text{if } y = 0 \end{cases}$$

We also show that $T_4 v \in U$. Moreover, we prove that $\Delta T_4 v(x,y) \geq -M\,c^l$.

$$\Delta T_4 v(x,y) = \begin{cases} y\Delta v(x,y-1) + (M-y)\Delta v(x,y) \geq -Mc^l & \text{if } y > 0 \\ M\Delta v(x,y) \geq -Mc^l & \text{if } y = 0 \end{cases}$$

$$\Delta^2 T_4 v(x,y) = \begin{cases} y\Delta^2 v(x,y-1) + (M-y)\Delta^2 v(x,y) \geq 0 & \text{if } y > 0 \\ M\Delta^2 v(x,y) \geq 0_l & \text{if } y = 0 \end{cases}$$

$$\Delta T_4 v(x,y+1) - \Delta T_4 v(x,y) = \begin{cases} (M-y-1)(\Delta v(x,y+1) - \Delta v(x,y)) & \text{if } y > 0 \\ \quad +y(\Delta v(x,y) - \Delta v(x,y-1)) \geq 0 & \\ (M-y-1)(\Delta v(x,y+1) - \Delta v(x,y)) \geq 0 & \text{if } y = 0 \end{cases}$$

$$\Delta T_4 v(x+1,y) - \Delta T_4 v(x,y+1) = \begin{cases} (M-y)(\Delta v(x+1,y) - \Delta v(x,y+1)) & \\ \quad +y(\Delta v(x+1,y-1) - \Delta v(x,y)) & \text{if } y > 0 \\ \quad +\Delta v(x,y+1) - \Delta v(x,y) \geq 0 & \\ M(\Delta v(x+1,y) - \Delta v(x,y+1)) & \text{if } y = 0 \\ \quad +(y+1)(\Delta v(x,y+1) - \Delta v(x,y)) \geq 0 & \end{cases}$$

As T is a positive linear combination of operators T_i, $i = 0, 1, \ldots, 4$, it comes that Tv satisfies conditions C.1, C.2, and C.3. To show that Tv satisfies C.4, it is slightly more tricky.

From optimality equations, we have:

$$\Delta Tv(x,y) = \frac{1}{C+\alpha}\begin{bmatrix} h + \mu\Delta T_0 v(x,y) + \lambda\Delta T_1 v(x,y) \\ +\delta\Delta T_2 v(x,y) + \gamma p\Delta T_3 v(x,y) \\ +\gamma p\Delta T_4 v(x,y) \end{bmatrix}$$

Using the fact that $\Delta T_i v \geq -c^l$ for $i = 0, 1, 2$ and $\Delta T_i v \geq -M\, c^l$ for $i = 3, 4$, we obtain

$$\Delta Tv(x,y) \geq \frac{-\mu c^l - \lambda c^l - \delta c^l - \gamma M p c^l - \gamma M q c^l}{C + \mu + \lambda + \delta + M\gamma} \geq -c^l$$

and Tv satisfies C.4. Finally, we conclude that Tv belongs to U.

C.2 Proof of Property 5.3.3

i. For problem A and production policy π, we adopt the following notations: rate of lost sales (λ^π_{uns}), rate of demands that are satisfied by produced items (λ^π_{prod}),rate of demands that are satisfied by returned items (λ^π_{ret}), average inventory level (I^π), average cost (g^π). Note that $\lambda^\pi_{ret} = p\delta$ is policy independent while the other quantities are policy dependent. For problem B and production policy π, we adopt similar notations with a bar. A demand is either lost or satisfied. When it is satisfied, it can be by a produced item or by a return. Therefore we have the following balance equation

$$\lambda = \lambda^\pi_{uns} + \lambda^\pi_{prod} + \lambda^\pi_{ret}$$

The average cost for instance A can then be related to the average cost for problem B:

$$\begin{aligned} g^\pi &= \lambda^\pi_{uns} c^l + \lambda^\pi_{prod}(\pi) c^m + \delta p c^r + h I^\pi \\ &= \lambda^\pi_{uns} c^l + (\lambda - \lambda^\pi_{uns} - \delta p) c^m + \delta p c^r + h I^\pi \\ &= \lambda^\pi_{uns}(c^l - c^m) + h I^\pi + \underbrace{\lambda c^m + \delta p (c^r - c^m)}_{=K>0} = \overline{g}^\pi + K \end{aligned}$$

Where K is a positive constant since, by assumption, $\lambda > p\delta$. Therefore, a policy minimizing g^π also minimizes $\overline{g}^\pi$.

ii. Follows directly from the proof of i. As $g^\pi = \overline{g}^\pi + K$ and the optimal policies are identical for A and B, we have immediately the desired relations between the average costs.

iii.

$$\begin{aligned}\Delta g &= \tfrac{g(noARI)-g(ARI)}{g(ARI)} \\ &= \tfrac{\overline{g}(noARI)+K-\overline{g}(ARI)-K}{\overline{g}(ARI)+K} \\ &\leq \tfrac{\overline{g}(noARI)-\overline{g}(ARI)}{\overline{g}(ARI)} = \Delta\overline{g}\end{aligned}$$

C.3 Proof of Theorem 5.5.1

The structure of the proof is similar to the one of Theorem 1. We have to prove that $\bar{T}$ propagates condition C.1 to C.6. We can prove it by combine the proof when only ARI is used (Lemma 1, Section 3) and the proof when only ADI is used (Lemma 1 in Gayon et al. (2009a)).

In the proof of Lemma 1, we show that operators T_0, T_2, T_3, T_4 propagate conditions C.1, C.2, C.3 and C.4. Gayon et al. (2009a) show that operators T_0, $\bar{T}_1$, T_5, T_6 propagate conditions C.1, C.4, C.5 and C.6. Remains to prove that operators T_2, T_3, T_4 propagate conditions C.5 and C.6, and that $\bar{T}_1$, T_5, T_6 propagate conditions C.2 and C.3. The proof of these propagation results is trivial because the state transitions involved in the operators are not involved in the conditions.

C.4 Source code

```c
#include <stdio.h>
#include <stdlib.h>

/* return the min/max between the two arguments */
double d_min(double a, double b){
  if (a<b) return a;
  else return b;
}
int    i_max(int    a,    int b){
  if (a>b) return a;
  else return b;
}

/* conversion from vector X to position in the vector v */
int X_to_x(int *X,int *lim, int *lim_dec,int dim){
  int i,j,x=0,prod;
  for(i=0;i<dim;i++){
    prod=1;
    for(j=0;j<i;j++) prod=prod*lim[j];
    x+=X[i]*prod;
```

```
  }
  return(x);
}

int main(){
  /* parametres of the system */
  double p=0.5,lambda=1,mu=1,delta=0.5,gamma=1;
  double ch=1,cl=2,cp=0,cr=0,alpha=0,precision=1e-6;
  int dim=2,lim[2]={20,20},lim_dec[2]={0,0},X_base[2]={0,0};
  /* system parameters and initialisation */
  double temp,max,min,final;
  int x,test=0,nb_sp=1;
  for(x=0;x<dim;x++) nb_sp=nb_sp*lim[i]; /* len of v */
  double *v=(double*)malloc(nb_sp*sizeof(double));
  double *vp=(double*)malloc(nb_sp*sizeof(double));
  for(x=0;x<nb_sp;x++) vp[x]=0;
  int x_base=X_to_x(X_base,lim,lim_dec,dim);
  int x0_high=lim[0]+lim_dec[0]-1,x1_high=lim[1]+lim_dec[1]-1;
  int x0_low=lim_dec[0],x1_low=lim_dec[1];
  int x0=x0_low,x1=x1_low;
  int x0P1=1,x1P1=lim[0];
  /* uniformisation rate */
  double C=alpha+lambda+mu+delta+x1_high*gamma;
  /* fixed point algorithm */
  while (test==0){
    for(x=0;x<nb_sp;x++){
      /* cost function */
      temp=i_max(x0,0)*ch;
      /* production operator */
      if (x0<x0_high)temp+=mu*d_min(vp[x],vp[x+x0P1]+cp);
      else temp+=mu*vp[x];
      /* demand operator */
      if(x0>x0_low)
        temp+=lambda*vp[x-x0P1];
      else temp+=lambda*(vp[x]+cl);
      /* return operator */
      if(x1<x1_high) temp+=delta*vp[x+x1P1];
      else temp+=delta*vp[x];
      /* ARI realized with probability p*/
      if(x1>x1_low && x0<x0_high)
```

```
            temp+=p*gamma*(x1*(vp[x+x0P1-x1P1]+cr)+(x1_high-x1)*vp[x]);
        else temp+=p*gamma*x1_high*vp[x];
        /* ARI not realized with probability (1-p)*/
        if(x1>x1_low)
            temp+=(1-p)*gamma*(x1*vp[x-x1P1]+(x1_high-x1)*vp[x]);
        else temp+=(1-p)*gamma*x1_high*vp[x];
        /* uniformisation */
        v[x]=temp/C;
        /* next state  */
        x0++;
        if(x0>x0_high){
            x1++;
            x0=x0_low;
            if(x1>x1_high)x1=x1_low;
        }
    }
    /* test of convergence */
    max=v[x_base]-vp[x_base];
    min=v[x_base]-vp[x_base];
    for(x=0;x<nb_sp;x++){
        temp=v[x]-vp[x];
        if (temp>max) max=temp;
        else if (temp<min) min=temp;
        vp[x]=v[x];
        if (alpha==0) vp[x]-=v[x_base];
    }
    if((alpha==0 && (max-min)/max<precision)
        ||(alpha!=0 && max < precision))
        test=1;
  }
  /* final = optimal cost */
  if(alpha==0) final=(min+max)/2*C;
  else final=v[x_base];
  printf("optimal cost\t%f\n",final);
  return 0;
}
```

Appendix D

Chapter 6

D.1 Boolean generalities

A Boolean value can take two values, $true = 1$ and $false = 0$. With a, b, c three Boolean value, we use two operators on them, the *and* "." and the *or* "+". The Table D.1 gives their truth tables and some of their properties are $a.(b+c) = (c+b).a = a.b+a.c = (a.b)+(a.c)$.

a	b	$a.b$	$a+b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Table D.1: Truth table of "." and "+"

D.2 Properties on the value function

Property D.2.1.

i) $|\Delta_{\mathbf{d}}\Omega_{A_i}v \geq 0| = |\Delta_{-\mathbf{d}}\Omega_{A_i}v \leq 0|$

ii) $\Delta_{\mathbf{d}}\Omega_{A_i}v(\mathbf{x}) = -\Delta_{-\mathbf{d}}\Omega_{A_i}v(\mathbf{x})$.

Proof.

$$\begin{aligned}
\Delta_{\mathbf{d}}\Omega_{A_i}v(\mathbf{x}) &= \Delta_{\mathbf{d}}(A_iv(\mathbf{x}) - v(\mathbf{x})) \\
&= -\Delta_{-\mathbf{d}}(A_iv(\mathbf{x}+\mathbf{d}) - v(\mathbf{x}+\mathbf{d})) \\
&= -\Delta_{-\mathbf{d}}\Omega_{A_i}v(\mathbf{x})
\end{aligned}$$

So $|\Delta_{\mathbf{d}}\Omega_{A_i}v(\mathbf{x}) \geq 0 \ \forall \mathbf{x}| = |\Delta_{-\mathbf{d}}\Omega_{A_i}v(\mathbf{x}+\mathbf{d}) \leq 0 \ \forall \mathbf{x}|$. $\square$

Property D.2.2. *With* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in (\mathbb{A} \cup \mathbb{E})^3$ *and* $(\mathbf{a}, \mathbf{b}) \in \mathbb{A}^2$.

i) $\mathtt{I}_{\boldsymbol{\alpha}} = \mathtt{D}_{-\boldsymbol{\alpha}}$

ii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathtt{S}^{ub}_{-\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathtt{S}^{ub}_{\boldsymbol{\alpha},-\boldsymbol{\beta}} = \mathtt{S}_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}$

iii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}.\mathtt{S}_{\boldsymbol{\gamma},\boldsymbol{\beta}} \geq \mathtt{S}_{\boldsymbol{\alpha}+\boldsymbol{\gamma},\boldsymbol{\beta}}$

iv) $|\Delta_{\boldsymbol{\alpha}} v \neq cte| \,.\, |\Delta_{\boldsymbol{\beta}} v \neq cte| \,.\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}.\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$

Proof. The property *i)* is true because $\Delta_{\mathbf{a}} v(\mathbf{x}) = \Delta_{-\mathbf{a}} v(\mathbf{x}+\mathbf{a})$. We prove the property *ii)* with the same argument: $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v(\mathbf{x}) = -\Delta_{-\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v(\mathbf{x}+\boldsymbol{\alpha}) = -\Delta_{\boldsymbol{\alpha}}\Delta_{-\boldsymbol{\beta}} v(\mathbf{x}+\boldsymbol{\beta}) = \Delta_{-\boldsymbol{\alpha}}\Delta_{-\boldsymbol{\beta}} v(\mathbf{x}+\boldsymbol{\alpha}+\boldsymbol{\beta})$. The property *iii)* is just the sum of two inequalities $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v(\mathbf{x}+\boldsymbol{\gamma}) \geq 0$ and $\Delta_{\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}} v \geq 0$. With simplification, $\Delta_{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}} v(\mathbf{x}) \geq 0$. Then, for the property *iv)* we suppose that v is not a linear function in direction $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. So $\Delta_{\boldsymbol{\alpha}} v \neq cte$ and $\Delta_{\boldsymbol{\beta}} v \neq cte$. It implies $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v \neq 0$. But the conditions $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ require $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v = 0$. □

D.3 Property on series

Property D.3.1. *Let a_i = and b_i be two series with $a_{i+1} \geq a_i \geq 0$, $b_{i+1} \geq b_i$ and $\sum_{i=0}^n b_i \geq 0$. Then $\sum_{i=0}^n a_i b_i \geq 0$.*

Proof. With $\bar{b} = \frac{1}{n}\sum_{i=0}^n b_i$, $\bar{b} \geq 0$. So

$$\sum_{i=1}^n a_i b_i = \underbrace{\sum_{i=1}^n a_i(b_i - \bar{b})}_{A} + \bar{b}\underbrace{\sum_{i=1}^n a_i}_{\geq 0}.$$

And with k such that

$$\begin{cases} b_i < \bar{b} \text{ if } i < k \\ b_i \geq \bar{b} \text{ otherwise} \end{cases}$$

$$\begin{aligned} A &= -\textstyle\sum_{i=0}^{k-1} \underbrace{a_i}_{\leq a_k}(\bar{b} - b_i) + \sum_{i=k}^n \underbrace{a_i}_{\geq a_k}(b_i - \bar{b}) \\ &\geq a_k \textstyle\sum_{i=0}^n (b_i - \bar{b}) = 0 \end{aligned}$$

□

D.4 Nested translation operator $T^t_{\mathbf{a},\mathbf{b}}$

With $\mathbf{y} = \mathbf{x} + \mathbf{b}$ and $\forall \mathbf{x}, \mathbf{x} + \mathbf{b} \in \mathbb{X}$,

$$T^t_{\mathbf{a},\mathbf{b}} v(\mathbf{x}) = \begin{cases} v(\mathbf{y}+\mathbf{a}) & \text{if } \mathbf{y}+\mathbf{a} \in \mathbb{X} \\ v(\mathbf{y}) + r & \text{else.} \end{cases}$$

D.4.1 Propagation of supermodularity

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v \geq 0$), then we want find conditions to have $T^t_{\mathbf{a},\mathbf{b}}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}T^t_{\mathbf{a},\mathbf{b}}v \geq 0$).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}T^t_{\mathbf{a},\mathbf{b}}v(\mathbf{x}) = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} \begin{cases} v(\mathbf{y}+\mathbf{a}) \text{ if } \mathbf{y}+\mathbf{a} \in \mathbb{X} \\ v(\mathbf{y}) + c \text{ else} \end{cases}$$

The four possible cases (A to D) are described in table D.2. Note that in this table, the cases E to P are used in following sections.

	$\mathbf{y}+\mathbf{a} \in \mathcal{S}$	$\mathbf{y}+\mathbf{a} \notin \mathcal{S}$
$\mathbf{y}+\mathbf{a}+\mathbf{d} \in \mathcal{S}$	A, E, I, and M	C, G, K, and O
$\mathbf{y}+\mathbf{a}+\mathbf{d} \notin \mathcal{S}$	B, F, J, and N	D, H, L, and P

Table D.2: cases

A and D are without any condition. Note that D is useless if $\mathtt{R}_{\mathbf{a},\mathbf{b}} + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})$. $B = \Delta_{\boldsymbol{\epsilon}}[v(\mathbf{y}+\mathbf{d})+r-v(\mathbf{y}+\mathbf{a})] = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a})+\epsilon_r$. So $B \geq 0$ if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_r \geq 0|$. Note that B is useless if $\mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d})$. $C = \Delta_{\boldsymbol{\epsilon}}[v(\mathbf{y}+\mathbf{d}+\mathbf{a})-v(\mathbf{y})-r] = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y})-\epsilon_r$. So $C \geq 0$ if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_r \leq 0|$. Note that C useless if $\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})$. In conclusion, a sufficient condition to have $T^t_{\mathbf{a},\mathbf{b}}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ is,

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_c \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d})).(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.|\epsilon_c \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}))$$

D.4.2 Marginal benefit $\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a},\mathbf{b}}}v \geq 0$

$$\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a},\mathbf{b}}}v(\mathbf{x}) = \Delta_{\mathbf{d}} \begin{cases} \Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) \text{ if } \mathbf{y}+\mathbf{a} \in \mathbb{X} \\ \Delta_{\mathbf{b}}v(\mathbf{x}) + r \text{ else} \end{cases}$$

The 4 cases are describe in Table D.2. The so the conditions to make the state "useless" are the same than previous section.

$E = \Delta_{\mathbf{d}}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})$. So $E \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}$. Note that E is independent of r.

$$\begin{aligned} F &= \Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{d}) + r - \Delta_{\mathbf{b}+\mathbf{a}}v(\mathbf{x}) \\ &= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x}) - \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}) + r \\ \Delta_{\mathbf{d}-\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{a}) - \Delta_{\mathbf{a}}v(\mathbf{x}) + r. \end{cases} \end{aligned}$$

So $F \geq 0$ if $|\Delta_{\mathbf{a}}v \leq r|.(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}})$. Note that F is increasing with r.

$$\begin{aligned} G &= \Delta_{\mathbf{b}+\mathbf{a}}v(\mathbf{x}+\mathbf{d}) - \Delta_{\mathbf{b}}v(\mathbf{x}) - r \\ &= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}+\mathbf{d}) - r \\ \Delta_{\mathbf{d}+\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{d}) - r. \end{cases} \end{aligned}$$

So $G \geq 0$ if $|\Delta_{\mathbf{a}} v \geq r| .(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}})$. Note that G is decreasing with r. $H = \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x})$. So $\mathtt{S}_{\mathbf{d},\mathbf{b}}$. Note that H is independent of r. In conclusion, a sufficient condition to have

- $\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a},\mathbf{b}}} v \geq 0$ is

$$\begin{aligned}
&\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}.(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}) + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}))\\
&.(|\Delta_{\mathbf{a}} v \leq r| .(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}) + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}))\\
&.(|\Delta_{\mathbf{a}} v \geq r| .(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}) + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}))
\end{aligned}$$

- $\Delta_{\epsilon_r}\Delta_{\mathbf{d}}\Omega_{T^t_{\mathbf{a},\mathbf{b}}} v \geq 0$ is $\mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})$

D.5 Nested choice operator $T^c_{\mathbf{a},\mathbf{b}}$

$$T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{y}), v(\mathbf{y}+\mathbf{a}) + c\} & \text{if } \mathbf{x}+\mathbf{a} \in \mathbb{X} \\ v(\mathbf{y}) + r, & \text{else} \end{cases}$$

with $\mathbf{y} = \mathbf{x} + \mathbf{b}$ and $\forall \mathbf{x}$, $\mathbf{x} + \mathbf{b} \in \mathbb{X}$.

D.5.1 Propagation of supermodularity

We make the assumption that $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}} = 1$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} v(\mathbf{x}) \geq 0$) then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $T^c_{\mathbf{a},\mathbf{b}}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} T^c_{\mathbf{a},\mathbf{b}} v \geq 0$).

The four possible cases of $\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}} T^c_{\mathbf{a},\mathbf{b}} v$ are the same than in table D.2.

$$\begin{aligned}
I &= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x})\\
&= \min\{v'(\mathbf{y}+\mathbf{d}), v'(\mathbf{y}+\mathbf{a}+\mathbf{d}) + c'\}\\
&\quad - \min\{v'(\mathbf{y}), v'(\mathbf{y}+\mathbf{a}) + c'\}\\
&\quad - \min\{v(\mathbf{y}+\mathbf{d}), v(\mathbf{y}+\mathbf{a}+\mathbf{d}) + c\}\\
&\quad + \min\{v(\mathbf{y}), v(\mathbf{y}+\mathbf{a}) + c\}
\end{aligned}$$

The 16 possible cases of I are described in Table D.3.

- $I_1 = \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a})$. without any condition.

- $I_2 = \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c \ = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c - \Delta_{\mathbf{a}} v'(\mathbf{y})$. So $I_2 \geq 0$ $|\epsilon_c \geq 0| .\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}$. Note that I_2 is useless if

$$\left\{ v(\mathbf{y}), v'(\mathbf{y}) | \forall \mathbf{y} : \begin{array}{c} \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) < -c \\ \Delta_{\mathbf{a}} v(\mathbf{y}) > -c \\ \Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) < -c' \\ \Delta_{\mathbf{a}} v'(\mathbf{y}) < -c' \end{array} \right\} = ,$$

	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \leq -c'$, $\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c'$	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \leq -c'$, $\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c'$	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \geq -c'$, $\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c'$	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \geq -c'$, $\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c'$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c$	I_1	I_5	I_9	I_{13}
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c$	I_2	I_6	I_{10}	I_{14}
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c$	I_3	I_7	I_{11}	I_{15}
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c$	I_4	I_8	I_{12}	I_{16}

Table D.3: cases

this condition is true if

$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \le c' \text{ so } \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0|\,. \end{cases}$$

- $I_3 = -\Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) + c \ge -\Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a})$. So $I_3 \ge 0$ without any condition.

- $I_4 = \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y})$
$$\ge \begin{cases} \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}) \\ \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \\ \qquad = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) \end{cases}$$
$I_4 \ge 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0|$. Note that I_4 is useless if $\Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v'(\mathbf{y})$ and $c \le c'$ so $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0|$.

- $I_5 = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + c' = \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + c' + \Delta_{\mathbf{a}} v'(\mathbf{y})$. So $I_5 \ge 0$ without condition.

- $I_6 = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c + c'$ So $I_6 \ge 0$ if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0|$. Note that I_6 is useless if $\Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d})$ so $\mathtt{S}_{\mathbf{d},\mathbf{a}}$.

- $I_7 = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + c + c'$. Note that I_7 is useless if
$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \ge c' \text{ so } \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \le 0| \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \le c' \text{ so } \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0| \end{cases}$$

- $I_8 = -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) + c' = -\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c'$. So $I_8 \ge 0$ if $|\epsilon_c \ge 0|\,.\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}$. Note that I_8 is useless if
$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \le c' \text{ so } \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0| \end{cases}$$

- $I_9 = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - c' = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c'$ So $I_9 \ge 0$ if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \le 0|$. Note that I_9 is useless if
$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \ge c' \text{ so } \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \le 0| \end{cases}$$

- $I_{10} = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + c' + c$ Note that I_{10} is useless if
$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \ge \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \ge c' \text{ so } \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \le 0| \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \le \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \le c' \text{ so } \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \ge 0| \end{cases}$$

- $I_{11} = -\Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) + c - c'$ So $I_{11} \geq 0$ if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|$. Note that I_{11} is useless if $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d})$ so $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$.

- $I_{12} = -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - c' = -\Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) - c$. So $I_{12} \geq 0$ without any condition.

- $I_{13} = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a})$
$$\geq \begin{cases} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) \\ \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \\ \qquad = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}). \end{cases}$$
So $I_{13} \geq 0$ if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|$. Note that I_{13} is useless if $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq \Delta_{\mathbf{a}} v'(\mathbf{y})$ and $c \leq c'$ so $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|$

- $I_{14} = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + c = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c$. So $I_{14} \geq 0$ without any condition.

- $I_{15} = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + c = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c$ So $I_{15} \geq 0$ if $|\epsilon_c \leq 0|\,.\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}$. Note that I_{15} is useless if
$$\begin{cases} \Delta_{\mathbf{a}} v(\mathbf{y}) \geq \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \text{ so } \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \\ \Delta_{\mathbf{a}} v(\mathbf{y}) \geq \Delta_{\mathbf{a}} v'(\mathbf{y}) \text{ and } c \geq c' \text{ so } \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| \end{cases}$$

- $I_{16} = -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y})$. So $I_{16} \geq 0$ without any condition.

Note that if $\Delta_a v \leq -c - \epsilon_c^+\}$ or $\Delta_a v \geq -c + \epsilon_c^-$ there is no condition because only cases I_1 and I_{16} could be reached.

In conclusion, a sufficient condition to have $I \geq 0$ is,

$$\begin{aligned}
&|\Delta_{\mathbf{a}} v \leq -c - \epsilon_c^+| + |\Delta_{\mathbf{a}} v \geq -c + \epsilon_c^-| + \\
&(|\epsilon_c \geq 0|\,.\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} + \mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|) && (I_2)\\
&.(\mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|) && (I_4)\\
&.(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0| + \mathtt{S}_{\mathbf{d},\mathbf{a}}) && (I_6)\\
&.(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|) && (I_7)\\
&.(|\epsilon_c \geq 0|\,.\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} + \mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|) && (I_8)\\
&.(|\epsilon_c \leq 0|\,.\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|) && (I_9)\\
&.(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d},\mathbf{a}} + \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0|) && (I_{10})\\
&.(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}) && (I_{11})\\
&.(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| + \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|) && (I_{13})\\
&.(|\epsilon_c \leq 0|\,.\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} + \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0|) && (I_{15})
\end{aligned}$$

With simplifications $I \geq 0$ if

$$|\Delta_{\mathbf{a}} v \leq -c - \epsilon_c^+| + |\Delta_{\mathbf{a}} v \geq -c + \epsilon_c^-|$$
$$+ \mathtt{S}_{\mathbf{d},\mathbf{a}}.\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \leq 0| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}.\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \geq 0|$$
$$+ \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}).\, |\epsilon_c = 0|$$

$$\begin{aligned} J &= \Delta_{\boldsymbol{\epsilon}}[T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x}+\mathbf{d}) - T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x})] \\ &= \Delta_{\boldsymbol{\epsilon}}[v(\mathbf{y}+\mathbf{d}) - T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x})] + \epsilon_r \end{aligned}$$

The cases to test are the same as I_{11}, I_{12}, I_{15}, and I_{16}. However properties $\mathtt{S}_{\mathbf{d},\mathbf{a}}$ and $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$ can not be used to make these cases useless.

- $I'_{11} = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + c - c'$. So $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \leq 0|$. I'_{11} useless if $|\Delta_{\mathbf{a}} v \geq -c|$.

- $I'_{15} = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c$. So $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \leq 0|$. I'_{15} useless if $|\Delta_{\mathbf{a}} v \geq -c|$.

- I'_{12} and I'_{16} are without any condition.

So $J \geq 0$ if $(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \leq 0| + |\Delta_{\mathbf{a}} v \geq -c|).\, |\epsilon_r \geq 0|$.

$$\begin{aligned} K &= \Delta_{\boldsymbol{\epsilon}}[T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x}+\mathbf{d}) - T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x})] \\ &= \Delta_{\boldsymbol{\epsilon}}[T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{y}+\mathbf{d}) - v(\mathbf{x})] - \epsilon_r \end{aligned}$$

In the same way, the cases to test are the same than I_6, I_8, I_{14}, and I_{16} (table D.3). However properties $\mathtt{S}_{\mathbf{d},\mathbf{a}}$ and $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$ can not be used to make these cases useless.

- $I'_6 = -\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - c + c'$. So $\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \geq 0|$. I'_6 useless if $|\Delta_a v \geq -c|$.

- $I'_8 = -\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c'$. So $\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \geq 0|$. I'_8 useless if $|\Delta_a v \geq -c|$.

- I'_{14} and I'_{16} are without any condition.

So $K \geq 0$ if $(\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\, |\epsilon_c \geq 0| + |\Delta_a v \geq -c|).\, |\epsilon_r \leq 0|$.

$$\begin{aligned} L &= \Delta_{\boldsymbol{\epsilon}}[T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x}+\mathbf{d}) - T^c_{\mathbf{a},\mathbf{b}} v(\mathbf{x})] \\ &= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) \end{aligned} .$$

So $L \geq 0$ without any condition.

In conclusion, a sufficient condition to have $T^c_{\mathbf{a},\mathbf{b}}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ is,

$$\begin{pmatrix} |\Delta_{\mathbf{a}} v \leq -c - \epsilon_c^+| + |\Delta_{\mathbf{a}} v \geq -c + \epsilon_c^-| \\ +\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},\mathbf{a}}.\,|\epsilon_c \leq 0| + \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \\ +\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.(\mathtt{S}_{\mathbf{d},-\mathbf{a}}.\,|\epsilon_c \geq 0| + \mathtt{S}_{-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c = 0|) \end{pmatrix}$$
$$.([\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \leq 0| + |\Delta_{\mathbf{a}} v \geq -c|]\,.\,|\epsilon_r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}))$$
$$.([\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}.\,|\epsilon_c \geq 0| + |\Delta_{\mathbf{a}} v \geq -c|]\,.\,|\epsilon_r \leq 0| + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}))$$

D.5.2 Marginal benefit $\Delta_d \Omega_{T^c_{\mathbf{a},\mathbf{b}}} v \geq 0$

The 4 possible cases of $\Delta_d \Omega_{T^c_{\mathbf{a},\mathbf{b}}} v(\mathbf{x})$ are the same as in Table D.2.

$$\begin{aligned} M &= \min\{v(\mathbf{y}+\mathbf{d}), v(\mathbf{y}+\mathbf{d}+\mathbf{a})+c\} - v(\mathbf{x}+\mathbf{d}) \\ &\quad - \min\{v(\mathbf{y}), v(\mathbf{y}+\mathbf{a})+c\} + v(\mathbf{x}) \end{aligned}$$

The 4 cases of M are described in Table D.4.

	$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c$	$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c$	M_1	M_3
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c$	M_2	M_4

Table D.4: cases

- $M_1 = \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x}) = \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x})$. So $M_1 \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}$. Note that M_1 is independent of c. Note that M_1 is useless if $|\Delta_{\mathbf{a}} v \geq -c|$.
- $M_2 = \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c \geq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$. So $M_2 \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$. Note that M_2 decreases with c. Note that M_2 is useless if $|\Delta_{\mathbf{a}} v \geq -c| + |\Delta_{\mathbf{a}} v \leq -c| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$.
- $M_3 = \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c \leq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$. Note that M_3 is useless if $|\Delta_{\mathbf{a}} v \geq -c| + |\Delta_{\mathbf{a}} v \leq -c| + \mathtt{S}_{\mathbf{d},\mathbf{a}}$.
- $M_4 = \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$. So $M_4 \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$. Note that M_4 is independent of c. Note that M_4 is useless if $|\Delta_{\mathbf{a}} v \leq -c|$.

So the sufficient condition to have,

- $M \geq 0$ is

$$\begin{aligned} &(\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} + |\Delta_{\mathbf{a}} v \geq -c|) && (M_1) \\ &.(\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}} v \geq -c| + |\Delta_{\mathbf{a}} v \leq -c| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}) && (M_2) \\ &.(|\Delta_{\mathbf{a}} v \geq -c| + |\Delta_{\mathbf{a}} v \leq -c| + \mathtt{S}_{\mathbf{d},\mathbf{a}}) && (M_3) \\ &.(\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}} v \leq -c|) && (M_4). \end{aligned}$$

With simplification, $M \geq 0$ if

$$\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}.\left|\Delta_{\mathbf{a}}v \leq -c\right| + \mathtt{S}_{\mathbf{d},\mathbf{b}}.\left|\Delta_{\mathbf{a}}v \geq -c\right| + \mathtt{S}_{\mathbf{d},\mathbf{b}}.\mathtt{S}_{\mathbf{d},\mathbf{a}}.$$

- With $\epsilon_c \geq 0$, $\Delta_{\epsilon_c} M \geq 0$ if $|\Delta_{\mathbf{a}}v \geq -c| + |\Delta_{\mathbf{a}}v \leq -c| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$.

$$N = v(\mathbf{y}+\mathbf{d}) + r - v(\mathbf{x}+\mathbf{d}) - T^c_{\mathbf{a},\mathbf{b}}v(\mathbf{x}) + v(\mathbf{x})$$

With $r \geq 0$, the cases are the same than M_2, and M_4. However property $\mathtt{S}_{\mathbf{d},\mathbf{a}}$ and $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$ can not be used to make these cases useless.

- $M'_2 = \Delta_{\mathbf{d}}v(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{x}) - \Delta_{\mathbf{a}}v(\mathbf{y}) - c \geq \Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}}v(\mathbf{x})$. So $\mathtt{S}_{\mathbf{d},\mathbf{b}}$ and M'_2 useless if $\Delta_{\mathbf{a}}v \geq -c$.

- $M'_4 \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$ and M'_4 is useless if $\Delta_a v \leq -c$.

If $\Delta_{\mathbf{a}}v \leq -c$ the condition is the same than case F for the marginal benefit of operator $T^t_{\mathbf{a},\mathbf{b}}$ $(|\Delta_{\mathbf{a}}v \leq r - c| .(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}))$.

So the condition to have $N \geq 0$ is

$$\begin{aligned}&|r \geq 0| .(\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}}v \geq -c|).(\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}}v \leq -c|)\\ &+ |\Delta_{\mathbf{a}}v \leq -c| . |\Delta_{\mathbf{a}}v \leq r - c| .(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}})\end{aligned}$$

Moreover, note that N increases with r, N independant of c if $\Delta_{\mathbf{a}}v \geq -c$ and N decreasing with c without condition.

$$O = T^c_{\mathbf{a},\mathbf{b}}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y}) - r + v(\mathbf{x})$$

With $r \leq 0$, the cases are the same than M_3, and M_4. However property $\mathtt{S}_{\mathbf{d},\mathbf{a}}$ and $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$ can not be used to make these cases useless.

- $M'_3 = \Delta_{\mathbf{d}}v(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{x}) + \Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) + c \leq \Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}}v(\mathbf{x})$. M'_3 useless if $\Delta_{\mathbf{a}}v \geq -c$.

- $M'_4 \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$ and M'_4 is useless if $\Delta_a v \leq -c$.

If $\Delta_a v \leq -c$ the condition is the same than case G for the marginal benefit of operator $T^t_{\mathbf{a},\mathbf{b}}$ $(|\Delta_{\mathbf{a}}v \geq r - c| .(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}))$. So the condition to have $O \geq 0$ is

$$\begin{aligned}&|r \geq 0| . |\Delta_{\mathbf{a}}v \geq -c| .(\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}}v \leq -c|)\\ &+ |\Delta_{\mathbf{a}}v \leq -c| . |\Delta_{\mathbf{a}}v \geq r - c| .(\mathtt{S}_{\mathbf{b},\mathbf{d}} + \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}).\end{aligned}$$

Note that O decreases with r, O independent of c if $\Delta_{\mathbf{a}}v \geq -c$ and O increasing with c without any condition.

$P = \Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})$. So $P \geq 0$ if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$. Note that P is independent of c.

In conclusion, sufficient conditions to have,

- $\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}}v \geq 0$ is,

$$\begin{gathered}(\mathtt{S}_{\mathbf{d},\mathbf{b}} + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}) + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d})) \\ .\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}}.\mathtt{S}_{\mathbf{d},\mathbf{b}} + |\Delta_{\mathbf{a}}v \geq -c|\,.\mathtt{S}_{\mathbf{d},\mathbf{b}} \\ + |\Delta_{\mathbf{a}}v \leq -c|\,.\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \end{pmatrix} \\ .\begin{pmatrix} |\Delta_{\mathbf{a}}v \leq \min\{-c, r-c\}|\,.(\mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} + \mathtt{S}_{\mathbf{b},\mathbf{d}}) \\ +\mathtt{S}_{\mathbf{d},\mathbf{b}}.\,|r \geq 0| + \mathtt{R}_{\mathbf{a},\mathbf{d},-\mathbf{b}}(\mathbf{a}+\mathbf{d}) \end{pmatrix} \\ .\begin{pmatrix} |\Delta_{\mathbf{a}}v \leq -c|\,.\,|\Delta_{\mathbf{a}}v \geq r-c|\,.(\mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}} + \mathtt{S}_{\mathbf{b},\mathbf{d}}) \\ +\mathtt{S}_{\mathbf{d},\mathbf{b}}.\,|r \leq 0|\,.\,|\Delta_{\mathbf{a}}v \geq -c| + \mathtt{R}_{\mathbf{a},-\mathbf{d},-\mathbf{b}}(\mathbf{a}-\mathbf{d}) \end{pmatrix}\end{gathered}$$

- $\Delta_{\epsilon_c}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}}v \geq 0$ is

$$|\Delta_{\mathbf{a}}v \geq -c| + (|\Delta_{\mathbf{a}}v \leq -c| + \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}).\mathtt{R}_{\mathbf{a},\mathbf{d}}$$

- $\Delta_{\epsilon_r}\Delta_{\mathbf{d}}\Omega_{T^c_{\mathbf{a},\mathbf{b}}}v \geq 0$ is $\mathtt{R}_{\mathbf{a},-\mathbf{d}}$

We can simplify these results because if $|\Delta_{\mathbf{a}} \leq -c|$ the choice $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is always chosen in the minimization, so the operator is equivalent to $T_{\mathbf{a},\mathbf{b},c}$, and if $|\Delta_{\mathbf{a}} \leq -c|$ the choice $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is never chosen in the minimization, so the operator is equivalent to $T_{\mathbf{0},\mathbf{b},0}$.

Note that without this simplification

$$\begin{gathered}\lim_{c\to-\infty}\left|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}(pT^c_{\mathbf{a},\mathbf{b}}v(\mathbf{x}) - c + p_0v(\mathbf{x})) \geq 0\ \forall\mathbf{x}\right| \\ = \left|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}(pT^t_{\mathbf{a},\mathbf{b}}v(\mathbf{x}) + p_0v(\mathbf{x})) \geq 0\ \forall\mathbf{x}\right|.\end{gathered}$$

We could predict it because the choice operator is a generalization of the translation operator. We force the choice with an infinite profit associated with the translation to $\mathbf{x}+\mathbf{a}+\mathbf{b}$.

D.6 Proof of Theorem 6.6.3

According to the Theorems 6.5.1 and 6.6.1 the condition to have $\mathcal{T}$ which propagates $\mathtt{S}_{\mathbf{e},\boldsymbol{\epsilon}}$ is

$$
\begin{aligned}
&|\Delta_{\mathbf{e}}\Delta_{\boldsymbol{\epsilon}}C \geq 0| \,. \bigotimes_{i\in\{0,1,2,3\}} |A_i \text{ propagates } \mathtt{S}_{\mathbf{e},\boldsymbol{\epsilon}}| \\
&.\left(|\epsilon_\mu < 0| \,.\mathtt{IMB}(-\mathbf{e},A_0) + |\epsilon_\mu > 0| \,.\mathtt{IMB}(\mathbf{e},A_0) + |\epsilon_\mu = 0|\right) \\
&.\bigoplus_{I_c^c\subset\{1,2,3\}} \left[\begin{array}{c}
\displaystyle\bigotimes_{i<j\in {I_c^c}^2} \left(\begin{array}{c}
\left|0 \leq \Delta_{\mathbf{e}}\Omega_{A_i}v \leq \Delta_{\mathbf{e}}\Omega_{A_j}v\right| . \left|\epsilon_{\lambda_i} \leq \epsilon_{\lambda_j}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \geq 0\right| \\
+\left|0 \leq \Delta_{\mathbf{e}}\Omega_{A_j}v \leq \Delta_{\mathbf{e}}\Omega_{A_i}v\right| . \left|\epsilon_{\lambda_j} \leq \epsilon_{\lambda_i}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \geq 0\right| \\
+\left|0 \leq \Delta_{-\mathbf{e}}\Omega_{A_i}v \leq \Delta_{-\mathbf{e}}\Omega_{A_j}v\right| . \left|\epsilon_{\lambda_i} \geq \epsilon_{\lambda_j}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \leq 0\right| \\
+\left|0 \leq \Delta_{-\mathbf{e}}\Omega_{A_j}v \leq \Delta_{-\mathbf{e}}\Omega_{A_i}v\right| . \left|\epsilon_{\lambda_j} \geq \epsilon_{\lambda_i}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \leq 0\right|
\end{array} \right) \\
.\displaystyle\bigotimes_{i\notin I_c^c} \left(|\epsilon_{\lambda_i} < 0| \,.\mathtt{IMB}(-\mathbf{e},A_i) + |\epsilon_{\lambda_i} > 0| \,.\mathtt{IMB}(\mathbf{e},A_i) + |\epsilon_{\lambda_i} = 0|\right)
\end{array} \right]
\end{aligned}
$$

Using Theorems 6.5.2 and 6.6.2 the previous equation becomes

$$
\begin{aligned}
&|\epsilon_h \geq 0| \,. \bigotimes_{i\in\{1,2,3\}} |\epsilon_{R_i} \leq 0| \,. \left(|\epsilon_\mu < 0| \,.\mathtt{C}_{\mathbf{e}}.\mathtt{I}_{\mathbf{e}} + |\epsilon_\mu > 0| \,.false + |\epsilon_\mu = 0|\right) \\
&.\bigoplus_{I_c^c\subset\{1,2,3\}} \left[\begin{array}{c}
\displaystyle\bigotimes_{i<j\in {I_c^c}^2} \left(\begin{array}{c}
false. \left|\epsilon_{\lambda_i} \leq \epsilon_{\lambda_j}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \geq 0\right| \\
\mathtt{C}_{\mathbf{e}}.\mathtt{R}_{\mathbf{e},-\mathbf{e}}. \left|\epsilon_{\lambda_j} \leq \epsilon_{\lambda_i}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \geq 0\right| \\
+false. \left|\epsilon_{\lambda_i} \geq \epsilon_{\lambda_j}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \leq 0\right| \\
+false. \left|\epsilon_{\lambda_j} \geq \epsilon_{\lambda_i}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \leq 0\right|
\end{array} \right) \\
.\displaystyle\bigotimes_{i\notin I_c^c} \left(|\epsilon_{\lambda_i} < 0| \,.false + |\epsilon_{\lambda_i} > 0| \,.\mathtt{C}_{\mathbf{e}} + |\epsilon_{\lambda_i} = 0|\right)
\end{array} \right]
\end{aligned}
$$

Knowing that v is $\mathtt{C}_{\mathbf{e}}$ and $\mathtt{I}_{\mathbf{e}}$ this equation reduces to

$$
\begin{aligned}
&|\epsilon_h \geq 0| \,. \bigotimes_{i\in\{1,2,3\}} |\epsilon_{R_i} \leq 0| \,. |\epsilon_\mu \leq 0| \\
&.\bigoplus_{I_c^c\subset\{1,2,3\}} \left(\bigotimes_{i<j\in {I_c^c}^2} \left|\epsilon_{\lambda_j} \leq \epsilon_{\lambda_i}\right| . \left|\sum_{k\in I_c^c}\epsilon_{\lambda_k} \geq 0\right| . \bigotimes_{i\notin I_c^c} |\epsilon_{\lambda_i} \geq 0| \right)
\end{aligned}
$$

D.7 Gamma operator

$$\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\Gamma^{T_i}v(\mathbf{x}) = \Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\gamma(\mathbf{x})\Omega_T v(\mathbf{x}) + \Delta_{\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x})$$

and

$$
\begin{aligned}
\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\gamma(\mathbf{x})\Omega_T v(\mathbf{x}) &= \Delta_{\mathbf{a}}(\gamma(\mathbf{x}+\mathbf{b})\Omega_T v(\mathbf{x}+\mathbf{b}) - \gamma(\mathbf{x})\Omega_T v(\mathbf{x})) \\
&= \Delta_{\mathbf{a}}([\gamma(\mathbf{x}) + \Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b}) - \gamma(\mathbf{x})\Omega_T v(\mathbf{x})) \\
&= \Delta_{\mathbf{a}}(\gamma(\mathbf{x})(\Omega_T v(\mathbf{x}+\mathbf{b}) - \Omega_T v(\mathbf{x})) + [\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b})) \\
&= \Delta_{\mathbf{a}}(\gamma(\mathbf{x})\Delta_{\mathbf{b}}\Omega_T v(\mathbf{x}) + [\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b})) \\
&= [\gamma(\mathbf{x}) + \Delta_{\mathbf{a}}\gamma(\mathbf{x})]\Delta_{\mathbf{b}}\Omega_T v(\mathbf{x}+\mathbf{a}) \\
&\qquad +[\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\gamma(\mathbf{x}) + \Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b}+\mathbf{a}) \\
&\qquad -\gamma(\mathbf{x})\Delta_{\mathbf{b}}\Omega_T v(\mathbf{x}) \\
&\qquad -[\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b}) \\
&= \gamma(\mathbf{x})\Delta_{\mathbf{b}}\Delta_{\mathbf{a}}\Omega_T v(\mathbf{x}) \\
&\qquad +[\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Delta_{\mathbf{a}}\Omega_T v(\mathbf{x}+\mathbf{b}) \\
&\qquad +[\Delta_{\mathbf{a}}\gamma(\mathbf{x})]\Delta_{\mathbf{b}}\Omega_T v(\mathbf{x}+\mathbf{a}) \\
&\qquad +[\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b}+\mathbf{a})
\end{aligned}
$$

so

$$
\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\Gamma^{T_i} v(\mathbf{x}) = \left(\begin{array}{l}
\gamma(\mathbf{x})\Delta_{\mathbf{b}}\Delta_{\mathbf{a}} T v(\mathbf{x}) \\
+[1-\gamma(\mathbf{x})]\Delta_{\mathbf{b}}\Delta_{\mathbf{a}} v(\mathbf{x}) \\
+[\Delta_{\mathbf{b}}\gamma(\mathbf{x})]\Delta_{\mathbf{a}}\Omega_T v(\mathbf{x}+\mathbf{b}) \\
+[\Delta_{\mathbf{a}}\gamma(\mathbf{x})]\Delta_{\mathbf{b}}\Omega_T v(\mathbf{x}+\mathbf{a}) \\
+[\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}\gamma(\mathbf{x})].\Omega_T v(\mathbf{x}+\mathbf{b}+\mathbf{a})
\end{array} \right)
$$

Résumé

De nombreux retours de produits dus au recyclage et à la réutilisation des déchets se développent dans le but de préserver les ressources naturelles limitées de notre planète. Ces nouveaux flux interagissant avec les flux de production traditionnels, il est important de les piloter de façon à satisfaire au mieux les demandes des clients et minimiser l'encours dans la chaîne logistique. Nos travaux s'inscrivent dans cette démarche. Nous nous plaçons dans un contexte où la capacité de production est limitée et nous considérons un problème opérationnel de gestion des stocks et de la production intégrant des flux de retours.

Nous modélisons trois problèmes de production et de stockage à temps continu, avec des capacités de production limitées, des délais aléatoires et des coûts linéaires. Le premier prenant en compte la probabilité qu'un produit puisse être réutilisé comme produit fini ou seulement comme produit semi-fini (par partie), le deuxième présentant un problème où la réutilisation d'un retour comme produit fini nécessite une étape de remise à neuf et le troisième modélisant un système où les clients préviennent à l'avance du renvoi potentiel de leurs produits. Outre la caractérisation des politiques optimales de gestion, une part importante de nos contributions réside dans l'évaluation des performances de différentes politiques heuristiques et l'étude de l'impact de la capacité de production sur celles-ci.

Enfin, nous nous servons dans tout ce document d'outils permettant la caractérisation des politiques optimales. La dernière partie de ce document vise à développer ces outils et à permettre l'étude de l'effet des paramètres d'un système formulé en processus de décision Markovien sur la politique optimale de celui-ci.

Abstract

Flows of returns due to recycling and reusing waste are developing in order to preserve the limited natural resources of our planet. These new flows interact with the traditional production flows. Therefore, in order to provide customers with the best service level and minimize the stock in the supply chain, the control of the return flows appears to be of highest importance. We address this problem by modeling a situation with a limited porduction capacity and we consider an operational production/inventory problem that incorporates flows of returns.

We model three continuous-time production/inventory problems with limited production capacities, random lead times, and linear costs. In the first problem we take into account the probability that a product can be reused as a finished product or only as semi-finished product (by parts), in the second problem we include a step of remanufacturing before reusing the returned product, and in the third problem we consider a system with product returns that are announced in advance by the customers. Apart from the caracterization of the optimal policies for these cases, the performance assessments of some heuristic policies and the study of the poduction capacity effect on these heuristic policies stand as main contributions.

Throughout this work we have used existing tools to characterize optimal policies for different Markov decision processes. The last chapter aims to improve these tools and enable us to study the influence of several system parameters on its optimal policy.